Modeling Convex Subsets of Points – Part 1

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based on recent and current joint
with several collaborators…

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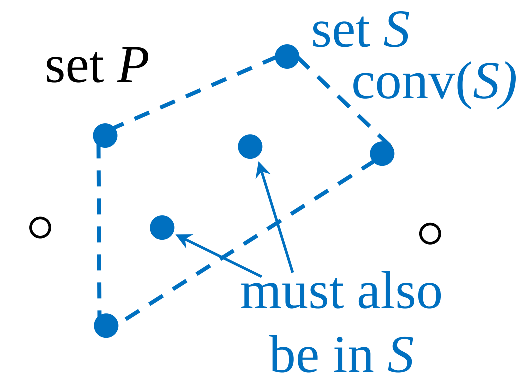
Convex Subsets of a Given Point Set

- A subset S of a given (finite) set P of n points in $\mathbf{R}^d$ is **convex** (relative to P) iff $S = P \cap \text{conv}(S)$ equivalently: $P \cap \text{conv}(S) \subseteq S$ (since $P \cap \text{conv}(S) \supseteq S$)
  - i.e., if we select a subset S of points in P then we must also select all points of P that are in their convex hull
  - definition also applies to more general *closure spaces* or *convexity spaces*



- We are interested in formulating the restriction "*the selected set of points must be convex*" in Integer Programming models
  - as hard or soft constraints

Why Convex Subsets?

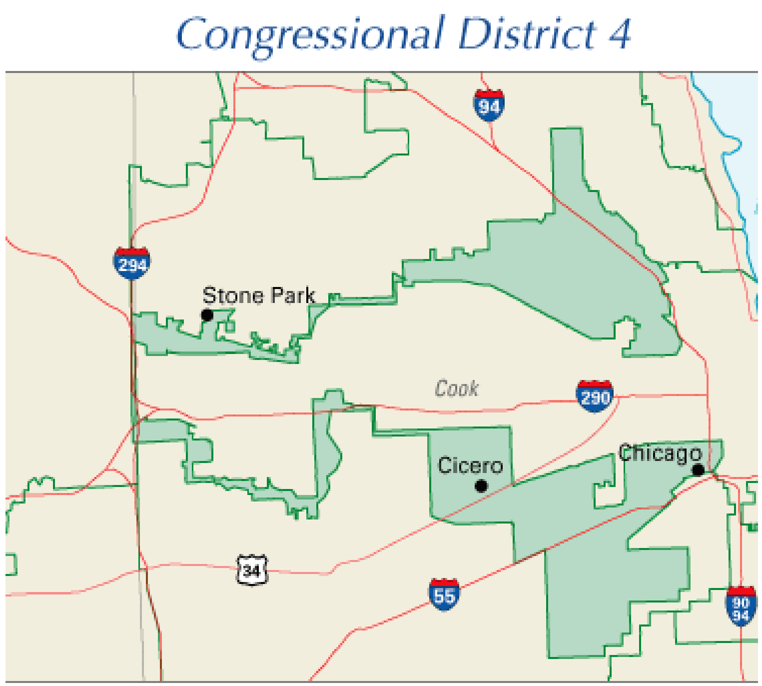
- Many (discrete) optimization models seek one or several subsets (of a given set of points, or of elementary regions, a.k.a., "cells") that should satisfy some "***shape constraint***"
  - often vaguely expressed: the set should "look compact", its shape should no be "too odd", etc.
- Convexity is one way of precisely formulating such shape constraints, which is appropriate (or approximately so) in some applications…

Why Convex Subsets? (2)

Some applications:

- Designing electoral districts
  - "Gerrymandering"

Gov. E. Gerry's "salamander"
Massachusetts, 1812



Illinois, 2004

- Douglas M. King, et al., 2012. Geo-Graphs: An Efficient Model for Enforcing Contiguity and Hole Constraints in Planar Graph Partitioning. *Operations Research* **60**(5) 1213–1228

Why Convex Subsets? (3)

More applications:

- Spatial Planning, e.g.,
  - Justin C. Williams, 2003. Convex land acquisition with zero - one programming. *Environment and Planning B: Planning and Design*, **30**, 255-270
- Farmland and Woodland Consolidation

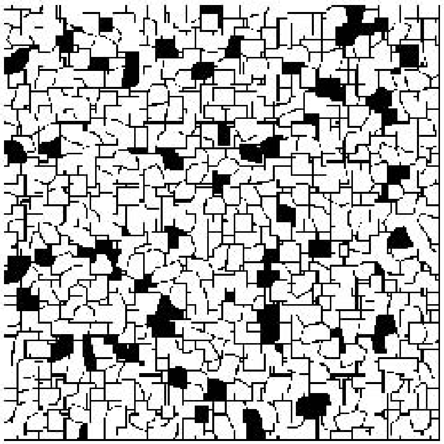
  (*Remembrement*, *Ruilverkaveling*)
  - Steffen Borgwardt, Andreas Brieden, and Peter Gritzmann, 2014. Geometric Clustering for the Consolidation of Farmland and Woodland. *Math. Intelligencer* **36** 37-44
- Police Quadrant Design
  - *work with Fernando Ordoñez and Flavio Guiñez (U. Chile)…*

Why Convex Subsets? (4)
Police Quadrant Design

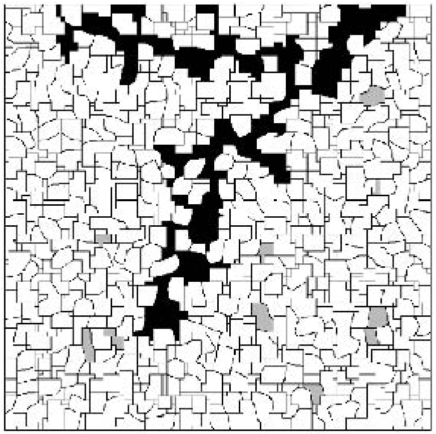
Why Convex Subsets? (5)

More applications:

- Forest Planning, e.g.,
  - M. Goycoolea, M., A. T. Murray, J.P. Vielma, A. Weintraub, 2009. Evaluating approaches for solving the area restriction model in harvest scheduling. *Forest Sci*. **55** (2) 149-165
  - Example: **old growth patch** from a Harvest Scheduling model
    - R. Carvajal, M. Constantino, M. Goycoolea, J.P. Vielma, A. Weintraub. Imposing Connectivity Constraints in Forest Planning Models, 2011



without shape constraints

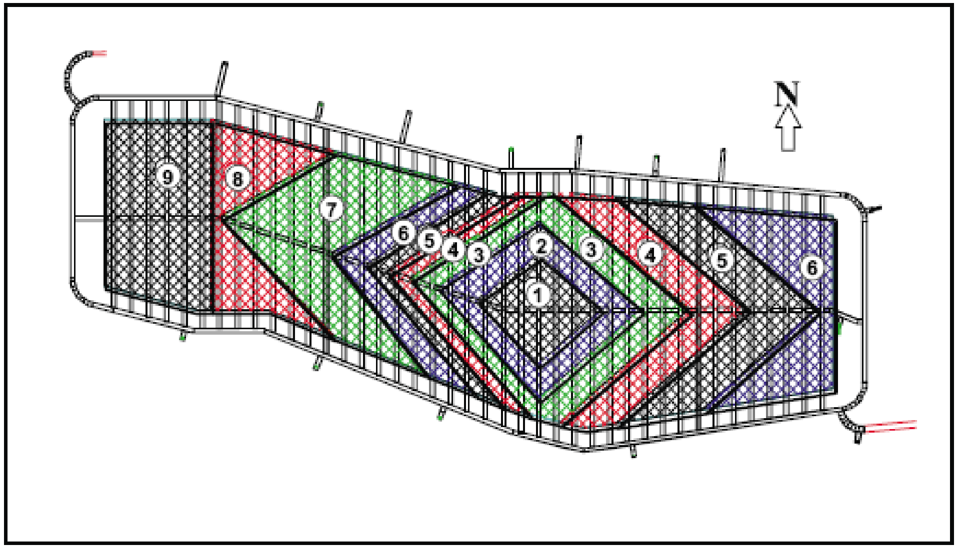


with connectivity constraint

Why Convex Subsets? (6)

More applications:

- Underground Mine Design and Scheduling, e.g.,
  - *PhD work of* ***Anita Parkinson*** *(UBC, 2012) with* ***Tony Diering*** *(Gemcom) and* ***Tom McCormick****, on drawpoint scheduling*
  - "convex" caves have better geomechanical stability
  - access constraints → 1D convexity along tunnels



current mine scheduling practice

Why Convex Subsets? (7)

Another 1-dimensional application:

- The Unit Commitment Problem in electric power generation
  - A generating unit must be "up" for successive time periods, i.e., for a convex subset of the planning horizon $P = \{1,2,\ldots,T\}$
  - Jon Lee, Janny Leung, François Margot, 2004. Min-up/min-down polytopes. *Discr. Opt.* **1** 77-85
  - Deepak Rajan, Samer Takriti, 2005. Minimum Up/Down Polytopes of the Unit Commitment Problem with Start-Up Costs. IBM Research Report.

Why Convex Subsets? (8)

…and more applications (in various dimensions):

- Data Mining, Statistical Clustering, Pattern Recognition, Data Compression, see, e.g., references in:
  - David Eppstein, Mark Overmars, Günter Rote, and Gerhard Woeginger, 1992. Finding Minimum Area k-gons. *Discrete and Computational Geometry* **7** 45-58
  - Paul Fischer, 1997. Sequential and parallel algorithms for finding a maximum convex polygon. *Computational Geometry* **7** 187-200.
  - C. Bautista-Santiago, J.M. Díaz-Báñez, D. Lara, P. Pérez-Lantero, J. Urrutia, and I. Ventura, 2011. Computing optimal islands. *Operations Research Letters* **39** (4) 246-251

Our Research Agenda

We seek to define notions of ("convex") shapes that are

- relevant to applications, and
- computationally tractable:
  - the Optimization Problem is efficiently solvable (or approximable)
  - the shape requirements can be enforced (or approximated) by a concise system of linear inequalities in natural and/or extended variables

Lectures Overview

Part 1: Computational complexity and algorithms

1. The Maximum Weight Convex Subset problem
2. Dimension 3 and higher: hardness results
3. One-dimension: a well understood case

Part 2: Modeling 2D and related convexities

1. 2D (points in the plane): DP algorithm for the optimization problem
2. 2D convex-shape constraints: IP modeling
3. Other notions of convexity
   a) Poset convexity
   b) Geodesic convexities and related notions

1. Maximum Weight Convex Subset Problem

To model convex-shape constraints with linear inequalities, the *Polytime Equivalence of Separation and Optimization* (GLS 1985) suggests considering the optimization problem:

- Given a set P of n points in $\mathbf{R}^d$ and **weights w_p** (of arbitrary sign) for all $p \in P$
- find a convex subset S of P with maximum weight $w(S) = \Sigma_{p \in S} w_p$

the **Maximum Weight Convex Subset problem**

- also of interest for Lagrangian Relaxation, and Column Generation

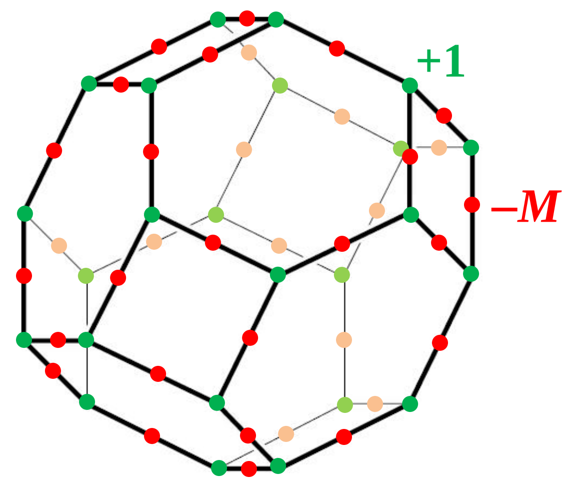
2. Dimension Three and Higher

- *Joint work with **Jeremy Barbay** (U.Chile), **Marek Chrobak** (U.C. Riverside) and **Miguel Constantino** (U. Lisboa)*

Theorem: The Maximum Weight Convex Subset problem is NP-hard for every dimension $d \geq 3$.

Proof: reduction from **MIS3CONPLAN**, the Maximum Independent Set problem on 3-connected planar graphs:

- embed instance $G = (V, E)$ of MIS3CONPLAN in $\mathbf{R}^3$ as the skeleton of a convex polytope with vertices $p(v)$ for all $v \in V$
- add the midpoint $p(u,v) = (p(u)+p(v))/2$ of every edge $(u,v) \in E$
- let weights $w_{p(u)} = 1$ and $$w_{p(u,v)} = -M < -|V|$$
- a subset S of these $|V|+|E|$ points is convex with $w(S) \geq 0$ iff $S \cap V$ is an independent set in G, and $w(S) = |S|$
- proof trivially extends to every dimension $d > 3$



QED

2. Dimension Three and Higher (2)

Theorem: When the dimension d is not fixed, the Maximum Weight Convex Subset problem cannot be approximated in polynomial time to within any factor $n^{1-\varepsilon}$ with $\varepsilon > 0$, unless P = NP

Proof: similar to previous proof, using the Maximum Independent Set (MIS) problem, known to be inapproximable to within any factor $n^{1-\varepsilon}$ with $\varepsilon > 0$, unless P = NP:

- embed instance $G = (V, E)$ of MIS in $\mathbf{R}^V$ where $p(v)$ is the v-th unit vector
- rest of the proof is identical
  - for every $(u,v) \in E$ add a point $p(u,v)$ at midpoint $(p(u)+p(v))/2$
  - let weights $w_{p(u)} = 1$ and $w_{p(u,v)} = -M < -|V|$
  - a subset S of these $|V| + |E|$ points is convex with $w(S) \geq 0$ iff $S \cap V$ is an independent set in G, and $w(S) = |S|$ **QED**

2. Dimension Three and Higher (3)

Theorem: The Maximum Weight Convex Subset problem is NP-hard for every dimension $d \geq 3$.

Theorem: When the dimension d is not fixed, the Maximum Weight Convex Subset problem cannot be approximated in polynomial time to within any factor $n^{1-\varepsilon}$ with $\varepsilon > 0$, unless P = NP

Open Problem: to find **approximation algorithms** (e.g., with constant factor, or PTAS, …) for the MWCS problem in dimension 3 (or higher)?

3. One-Dimension: a Well Understood Case

Given set P of n points $p_1 < p_2 < \ldots < p_n$ in $\mathbf{R}$, a subset $S \subseteq P$ is convex iff it consists of consecutive (or contiguous) points in P, i.e., iff $S = \{p_i, \ldots, p_j\}$ for some $i \leq j$ (or $S = \emptyset$)

- given any weights $w_1, \ldots, w_n$ the optimization problem (MWCS) is very easy
  - solved in linear time (by Dynamic Programming)
- with "natural" membership binary variables $y_j = 1$ iff $p_j \in S$ $O(n^3)$ **3-point constraints**

$$y_i \geq y_h + y_j - 1 \quad \text{for all } h < i < j$$

  suffice to enforce convexity of S (with binary variables)
  - Using $i = h+1$ suffices, so $O(n^2)$ constraints suffice

3. One-Dimension (2)

- … but the polyhedron defined by the 3-point alternating constraints and $0 \leq y \leq 1$ has fractional extreme points
- General **alternating constraints**:

$$\Sigma_{i=1,..,k} \ (-1)^{i+1} \ y_{d(i)} \ \leq \ 1 \ \text{ for every}$$

  - **odd** integer $k = 3, 5, \ldots \ (3 \leq k \leq n)$
  - (odd) subsequence $d(1) < d(2) < \ldots < d(k)$ of points

- Polyhedral result (Groeflin & Liebling, 1981; Lee & al., 2004):

  Theorem: In one dimension, the general alternating constraints and $0 \leq y \leq 1$ define the **convex hull** (in $\mathbf{R}^n$) of the set of all characteristic vectors y of contiguous subsets.

3. One-Dimension (3)

- General **alternating constraints:**

  $\Sigma_{i=1,..,k}\ (-1)^{i+1}\ y_{d(i)} \leq 1$ for every

  - **odd** integer $k = 3, 5, \ldots\ (3 \leq k \leq n)$ and
  - (odd) subsequence $d(1) < d(2) < \ldots < d(k)$
- About $2^{n\text{-}1}$ such constraints
- Cutting plane approach

  …requires solving the **Separation Problem**:

  Given vector y satisfying $0 \leq y \leq 1$

  - does y satisfy all alternating constraints?
  - and, if not, identify a violated constraint.
- O(n) exact separation algorithm (Lee & al.)

3. One-Dimension (4)

Extended Formulation:

- Auxiliary variable $F_i = 1$ indicates that point $p(i)$ is the **leftmost point** of the selected region S

$F_i \le y_i$	Point $p(i)$ must be selected to be leftmost
$F_i \ge y_i - y_{i-1}$	If $p(i)$ is selected but $p(i-1)$ is not, then $p(i)$ must be leftmost
$\sum_{i=1}^{n} F_i \le 1$	At most one leftmost point

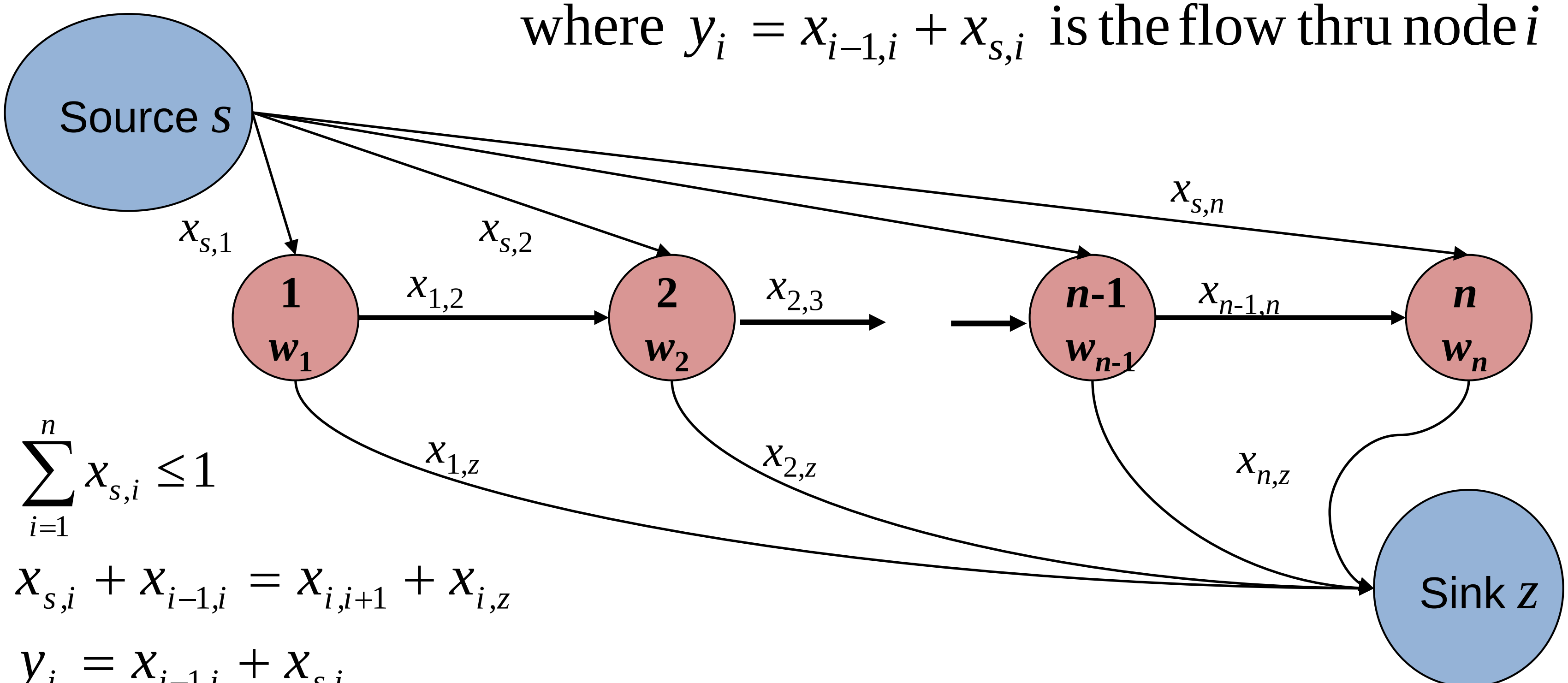
Theorem (Malkin & Wolsey, 2003; Rajan & Takriti,'05): This extended formulation is **ideal**, i.e., the extreme points of the corresponding polyhedron are the 0-1 vectors representing all the contiguous solutions

3. One-Dimension (5)

Formulation as a **Minimum Cost** (actually, Maximum Profit) **Network Flow Problem**:

- send at most 1 unit of flow to maximize $\sum_{i=1}^{n} w_i y_i$

where $y_i = x_{i-1,i} + x_{s,i}$ is the flow thru node i



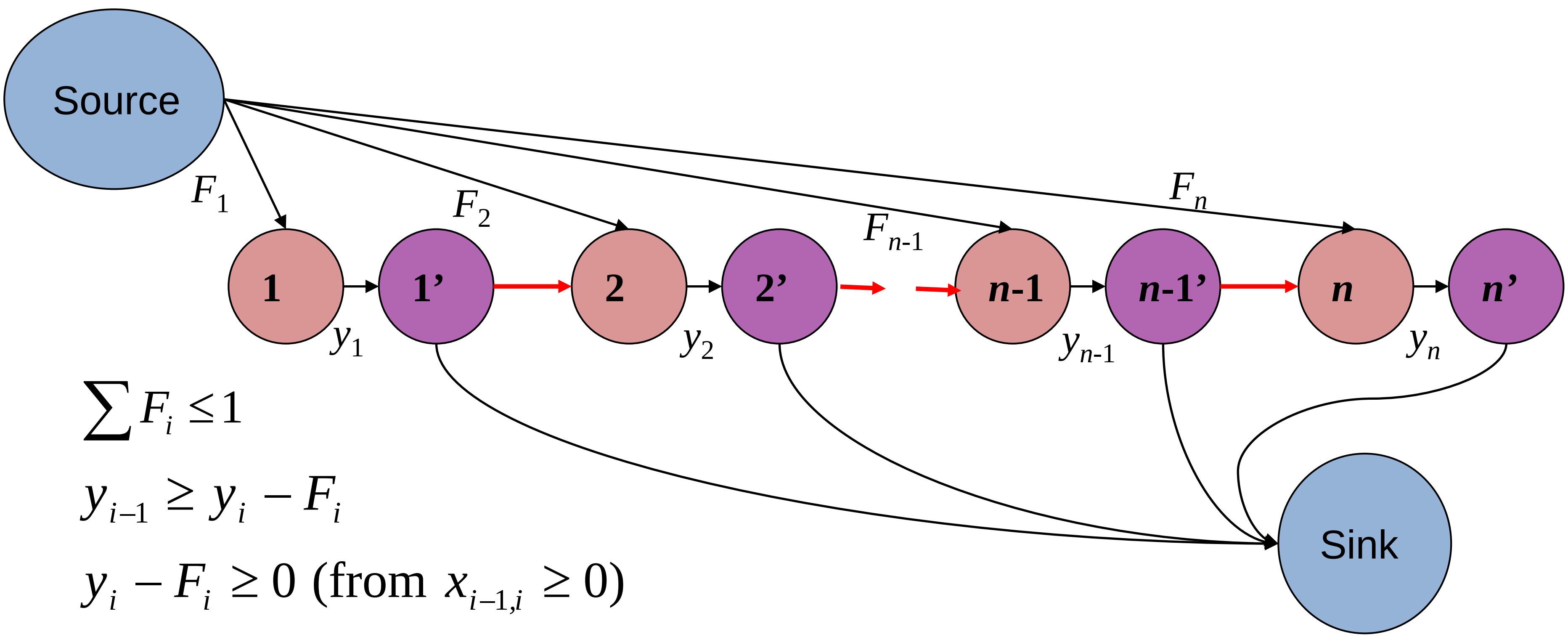
$$\sum_{i=1}^{n} x_{s,i} \leq 1$$

$$x_{s,i} + x_{i-1,i} = x_{i,i+1} + x_{i,z}$$

$$y_i = x_{i-1,i} + x_{s,i}$$

3. One-Dimension (6)

- Split each node to represent the flow through this node
- Extreme point solutions are integral
- Eliminate arc flow variables $x_{i\text{-}1,i}$ → Extended Formulation



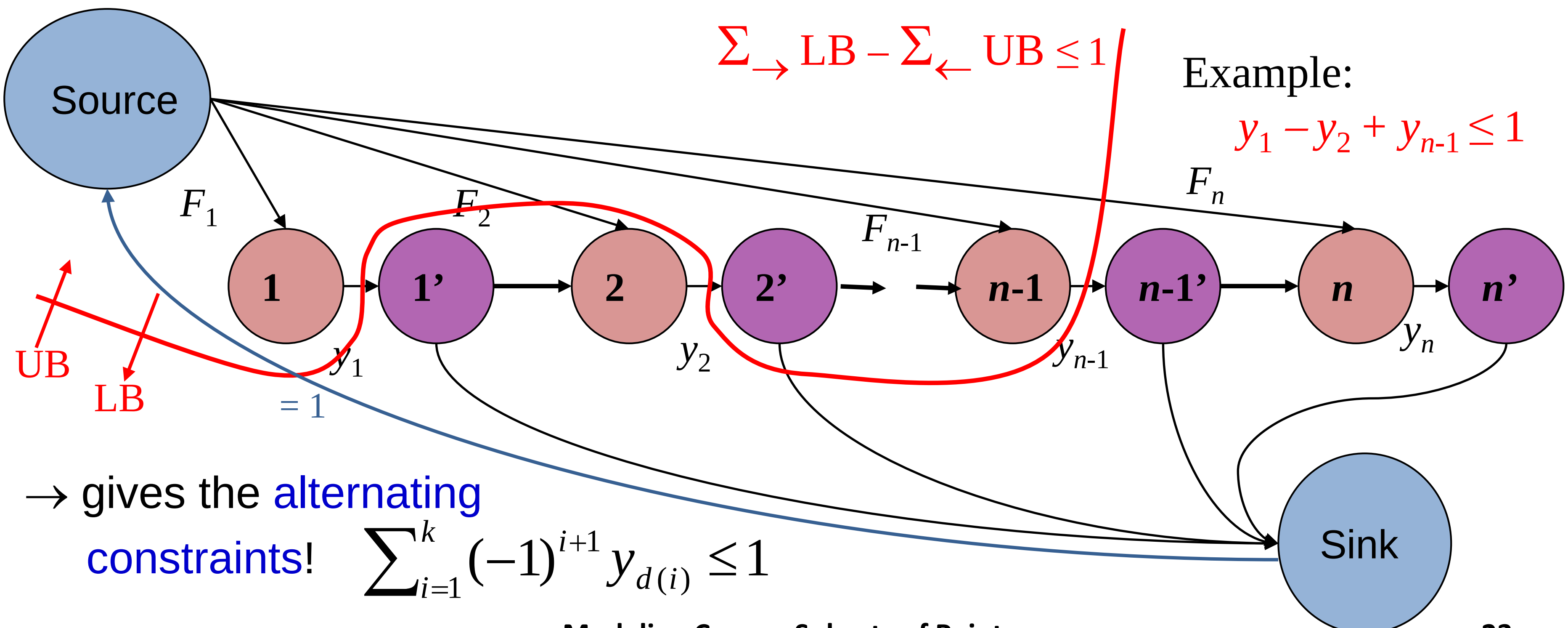
$$\sum F_i \leq 1$$

$$y_{i-1} \geq y_i - F_i$$

$$y_i - F_i \geq 0 \text{ (from } x_{i-1,i} \geq 0\text{)}$$

3. One-Dimension (7)

- Formulate Separation Problem for given vector y as feasibility flow with flow value 1; LB & UB y_i on split-node arcs, 0 & $+\infty$ elsewhere
- Apply Alan **Hoffman's *Feasible Circulation Theorem***:
  – finite capacity cuts $\leftrightarrow$ odd subsets $d(1) < d(2) < \ldots < d(k)$



→ gives the alternating constraints! $\sum_{i=1}^{k} (-1)^{i+1} y_{d(i)} \leq 1$

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