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MINLPs with few integer variables

ETH Zürich

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What and why?

MINLP model

Let $P \subseteq \mathbf{R}^{n+d}$ be a polytope and $f : \mathbf{R}^{n+d} \to \mathbf{R}$ a nonlinear function.

$$\begin{aligned} \min \quad & f(x, y) \\ \text{s.t.} \quad & \\ & (x, y) \in P, \\ & x \in \mathbf{Z}^d, \; y \in \mathbf{R}^n. \end{aligned}$$

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Why study this model?

- (MILP) and (CO) are about to become a technology.
- understand specific class of MINLPs: optimization over continous relaxations is "tractable".
- build a bridge to other areas of mathematics.

The central question

Can we extend theory and algorithms from MILP and NLO to the mixed integer setting?

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What do we aim at?

- Complexity results.
- Algorithmic schemes amenable to an analysis.

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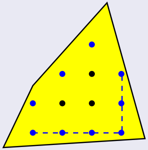
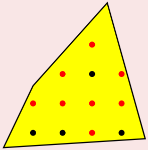
Aspects of nonlinear discrete optimization

Convex maximization



Polynomial optimization



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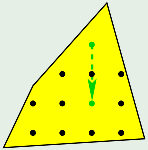

Parametric non-linear optimization and W-mappings

A borderline case from the point of view of computational complexity

$$\begin{array}{ll} \min & f(Wx) \\ \text{s.t.} & x \in P \cap \mathbf{Z}^n \end{array}$$

with $W \in \mathbf{Z}^{m \times n}$ where n is regarded as variable, but m as fixed.
("Maps variable dimension to fixed dimension" [Onn, Rothblum '05])

The landscape of computational complexity

Objective function	Variables		
	Dimension two	Fixed dimension	Parametric
Convex max	poly-time	poly-time	poly-time NP-hard
Convex min	poly-time	poly-time	poly-time NP-hard
Polynomial	poly-time NP-hard	FPTAS NP-hard	? ?

Concave minimization or convex maximization

Observation

Let P be a rational polytope in $\mathbf{R}^n$, and let f be such that for every $\bar{z} \in P \cap \mathbf{Z}^n$, the set $\{z \in P \mid f(z) \geq f(\bar{z})\}$ is convex. For fixed n, $\min\{f(x) \mid x \in P \cap \mathbf{Z}^n\}$ can be solved in polynomial time.

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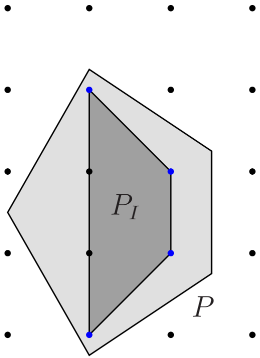
Proof

- We can find the set V of the vertices of P_I (Cook, Hartman, Kannan, McDiarmid, 1992)



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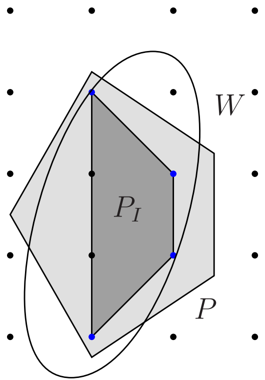
Proof

- We can find the set V of the vertices of P_I (Cook, Hartman, Kannan, McDiarmid, 1992)
- Let $\bar{z}$ be the best vertex, and let $W := \{z \in P \mid f(z) \geq f(\bar{z})\}$
- $V \subseteq W$
- As W is convex, $P_I = \operatorname{conv} V \subseteq W$



State of the art for convex minimization: fixed dimension

Theorem [Lenstra '83] [Grötschel, Lovász, Schrijver '88]

For any fixed $n \geq 1$, there exists an oracle-polynomial algorithm that, for any convex set $K \subseteq \mathbf{R}^n$ with $B(*, r) \subseteq K \subseteq B(0, R)$ given by a weak separation oracle, and for any rational $\varepsilon > 0$, either finds a point in $(K + B(0, \varepsilon)) \cap \mathbf{Z}^n$, or concludes that $K \cap \mathbf{Z}^n = \emptyset$.

Theorem [Khachiyan, Porkolab '00] and improvements by [Heinz '05], [Hildebrand, Köppe '12] [Dadush '13]

Let $g_1, \ldots, g_m \in \mathbf{Z}[x_1, \ldots, x_n]$ be quasi-convex polynomials of degree at most δ whose coefficients have a binary encoding length of at most s. There exists an algorithm for testing feasibility of

$$g_1(x) \leq 0, \ldots, g_m(x) \leq 0, \; x \in \mathbf{Z}^n.$$

whose running time is polynomial in m, s, δ provided that n is constant.

A general scheme for mixed integer convex minimization [Baes, Oertel, Wagner, W.] [Yudin, Nemirovskii 79]

An augmentation oracle

For a mixed integer set $\mathcal{F}$ and $x \in \mathbf{R}^n$ either **(a)** return a point $\hat{x} \in \mathcal{F}$ such that $f(\hat{x}) \leq (1+\alpha)f(x) + \delta$ or **(b)** assert non-existence.

Gradient Descent method (GDM) ($N \in \mathbf{Z}_+$, $x_0 = \hat{x_0} \in \mathcal{F}$)

For $k = 0, \ldots, N-1$, perform the following steps:

- **Determine** $x_{k+1} = x_k - h_k \nabla f(x_k)$
- **If** $f(x_{k+1}) \geq f(x_k)$ set $x_{k+1} = x_k$, $\hat{x_{k+1}} = \hat{x_k}$ and continue.
- **If** $f(x_{k+1}) < f(x_k)$ query the oracle with input x_k.
  - **If** the oracle output is **(a)**, then update $\hat{x_{k+1}}$.
  - **If** the oracle output is **(b)**, then start gap closing :
    For $l \leq f^* \leq u$ and precision $\epsilon > 0$, find $x \in \mathcal{F}$ such that

$$f(x) - f^* \leq \epsilon.$$

Analysis and extensions [Baes, Oertel, Wagner, W.]

Theorem. For $\alpha = \delta = 0$ and f convex with Lipschitz-constant L:

If (GDM) does not terminate before N steps, then

$$f(x_{best}) - f^* \leq L\sqrt{\frac{\delta_{\mathcal{F}}}{2}} \quad \frac{ln(N) + 2}{2\sqrt{N+2} - 2}.$$

The gap-closing algorithm can be implemented to run in oracle polynomial time in $ln(\epsilon)$ and in $ln(f(x_{best}) - f^*)$.

Extensions

- We can generalize GDM to Mirror-Descent Methods, for better convergence properties.
- Constrained problems: we need a projector and a separator from the continuous feasible set.
- We allow for $\alpha, \delta > 0$, without accumulation of errors during the iterations (smallest affordable gap: $(2 + \alpha)(\alpha\hat{f}^* + \delta)$).

Implementation of the oracle: optimality conditions

The continuous case without constraints

Theorem. Let f be convex and continously differentiable on its domain. Let $x^* \in$ dom f. Then, x^* attains the value

$$\min\{f(x) \mid x \in \text{ dom } f\}$$

if and only if

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The unconstrained mixed integer case: [Baes, Oertel, W.]

Theorem. Let $f : \mathbf{R}^{n+d} \mapsto \mathbf{R}$ be a continuous convex function. Then, $x^* \in \mathbf{Z}^n \times \mathbf{R}^d$ attains the value

$$\min\{f(x) \mid x \in \text{ dom } f, \; x \in \mathbf{Z}^n \times \mathbf{R}^d\}$$

if and only if there exist $k \leq 2^n$ points $x_1 = x^\star, x_2, \ldots, x_k \in \mathbf{Z}^n \times \mathbf{R}^d$ and vectors $h_i \in \partial f(x_i)$ such that the following conditions hold:

(a) $f(x_1) \leq \ldots \leq f(x_k)$,

(b) $\{x \mid h_i^T(x - x_i) < 0 \; \forall i\} \cap (\mathbf{Z}^n \times \mathbf{R}^d) = \emptyset$,

(c) $h_i \in \mathbf{R}^n \times \{0\}^d$ for $i = 1, \ldots, k$.

Implementation of the oracle: duality

Assumptions

Let $f : \mathbf{R}^{n+d} \mapsto \mathbf{R}$ and $g : \mathbf{R}^{n+d} \mapsto \mathbf{R}^m$ be differentiable, convex functions, $\emptyset \neq \{x \in \mathbf{R}^{n+d} \mid g(x) \leq 0\} \subset \operatorname{dom} f$ is compact. Let g fulfill the *(mixed-integer) Slater condition*.

Continuous Lagrangian duality

$$f^\star = \min_{x \in \mathbf{R}^n} \{ f(x) \mid g(x) \leq 0\} = \max_{\alpha, u \in \mathbf{R}^m_+} \{\alpha \mid \alpha \leq f(x) + u^T g(x) \forall x \in \mathbf{R}^n\}.$$

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Mixed integer duality [Baes, Oertel, W. 2014]

$$\begin{aligned} & \min_{x \in \mathbf{Z}^n \times \mathbf{R}^d} \{ f(x) \mid g(x) \leq 0\} \\ = \; & \max_{\substack{\alpha \in \mathbf{R} \\ U \in \mathbf{R}_+^{2^n \times m}}} \{ \alpha \mid \exists \, \pi : \mathbf{Z}^n \times \mathbf{R}^d \mapsto \{1, \ldots, 2^n\} \text{ s.t.} \\ & \forall x \in \mathbf{Z}^n \times \mathbf{R}^d \; \alpha \leq f(x) + U_{\pi(x)} g(x) \text{ or } 1 \leq U_{\pi(x)} g(x)\}. \end{aligned}$$

Behind duality: Mixed integer KKT theorem

KKT theorem under standard assumptions

$x^\star$ such that $g(x^\star) \leq 0$ attains the optimal solution if and only if there exist $h_f \in \partial f(x^\star)$, $h_{g_i} \in \partial g_i(x^\star)$, $\lambda_i \geq 0$ for all i such that

$$h_f + \sum_{i=1}^m \lambda_i h_{g_i} = 0 \text{ and } \lambda_i g_i(x^\star) = 0 \ \forall i.$$

Mixed integer version of KKT [Baes, Oertel, W. 2014]

$x^\star \in \mathbf{Z}^n \times \mathbf{R}^d$, $g(x^\star) \leq 0$ solves mixed integer convex problem if and only if there exist $k \leq 2^n$ points $x_1 = x^\star, x_2, \ldots, x_k \in \mathbf{Z}^n \times \mathbf{R}^d$ and k vectors $u_1, \ldots, u_k \in \mathbf{R}_+^{m+1}$ with $h_{i,m+1} \in \partial f(x_i)$, and $h_{i,j} \in \partial g_j(x_i)$ $\forall j$ such that:

(a) If $g(x_i) \leq 0$ then $f(x_i) \geq f(x_1)$, $u_{i,m+1} > 0$ and $u_{i,j} g_j(x_i) = 0 \ \forall j$,

(b) If $g(x_i) \not\leq 0$ then $u_{i,m+1} = 0$ and $u_{i,k}(g_k(x_i) - g_l(x_i)) \geq 0 \ \forall k, l$,

(c) $\{x \mid \sum_{j=1}^{m+1} u_{i,j} h_{i,j}^T (x - x_i) < 0 \text{ for all } i\} \cap (\mathbf{Z}^n \times \mathbf{R}^d) = \emptyset$,

(d) $\sum_{j=1}^{m+1} u_{i,j} h_{i,j} \in \mathbf{R}^n \times \{0\}^d$ for $i = 1, \ldots, k$.

Implementation of the oracle II: [Oertel, Wagner, W.]

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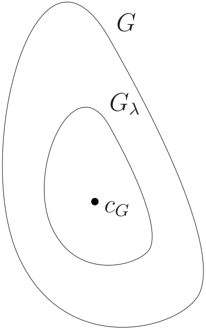
- Replace the ellispoid type method by a polytope shrinking algorithm.

The ingredients:

- For convex compact G, the centroid $c_G = \frac{\int_G x dx}{vol(G)}$.
- $G_\lambda := \lambda(G - c_G) + c_G$.



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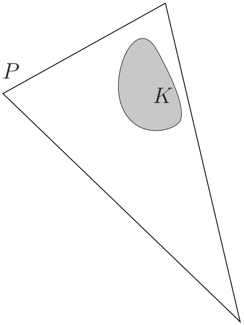

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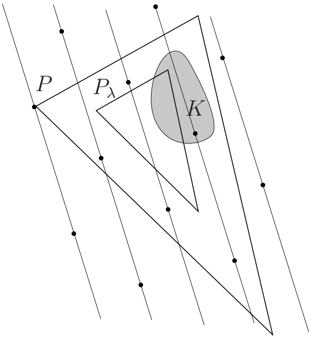

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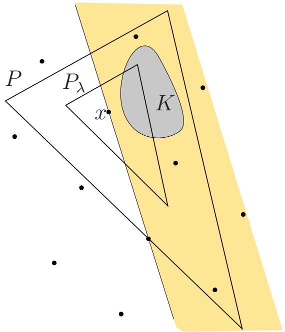
  If $x \notin K$, separate x.



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Extension of a theorem of Grünbaum 1960 ($\lambda = 0$)

Let G be a compact convex set, let H be a halfspace and let $0 < \lambda < 1$. If $G_\lambda \cap H \neq \emptyset$, then

$$\frac{\text{vol }(G \cap H)}{\text{vol }(G)} \geq (1-\lambda)^n \left(\frac{n}{n+1}\right)^n.$$

Analysis of the polytope-shrinking algorithm:

Iterations k until $\text{vol}(P) \leq \frac{1}{n!}$:

$$k \leq \frac{n\left[log(2B) + log(n)\right]}{(1-\lambda)^n(\frac{n}{n+1})^n}.$$

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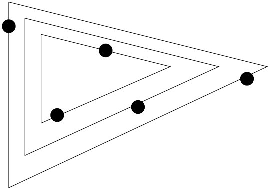

Good news about the computation of $x \in P_\lambda \cap \mathbf{Z}^n$:

- For n fixed, P_λ can be efficiently computed by solving a mixed integer linear program in dimension $n+1$:

$$\begin{aligned} t^* \; &= \max \; t \\ & a_i^T x + \omega(\mathbf{P}, \mathbf{a_i}) t \leq b_i \; \forall i \\ & x \in \mathbf{Z}^n, \; t \geq 0. \end{aligned}$$

- (x^*, t) feasible implies (a) $x^* \in P_{1-t}$ and
- (x^*, t) feasible implies (b) $x^* \in \{x \mid x + t(P-P) \subseteq P\}$.

W-mappings

The setting: $\min \{f(Wx) \ : \ Ax \leq b, \, x \in \mathbf{Z}^n\}$

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Why and what?

- Why do we need these oracles?
- Under which conditions on the input is this problem tractable?

Assumptions about $\min f(Wx)$ subject to $x \in \mathcal{F}$.

W is in unary representation.

We can model the Partition Problem:
For $w_1, \cdots, w_n \in \mathbf{Z}_+$ and
$D = \frac{1}{2}\sum_{i=1}^n w_i$, solve

$$\begin{array}{rl} \min & (w^T x - D)^2 \\ \text{s.t.} & x \in \{0,1\}^n. \end{array}$$

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Theorem [Lee, Onn, W. '10]
There is a universal constant ρ such that no polynomial time algorithm can compute a ρn-best solution of the nonlinear optimization problem $\min\{f(Wx) : \ x \in \mathcal{F}\}$ over any independence system $\mathcal{F}$ presented by a linear optimization oracle, not even with W a fixed integer $2 \times n$ matrix.

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The tractability question:

Conditions on $\mathcal{F}$ and A, b, resp. ?

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- Let $A \in \mathbf{Z}^{m \times n}$, $W \in \mathbf{Z}^{d \times n}$, $b \in \mathbf{Z}^m$ and $f : \mathbf{R}^d \to \mathbf{R}$.

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Theorem [Adjiashvili, Oertel, W. '14]

There is an algorithm that solves the non-linear optimization problem

$$\min \{ f(Wx) \,:\, Ax \leq b,\ x \in \mathbf{Z}^n \}.$$

The number of calls of the optimization and fiber oracles is polynomial in n and $\mathbf{\Delta}$.

Special properties on $\mathcal{F}$ make the problem tractable.

A first polynomial time algorithm.

Let $\mathcal{F} = \left\{x \in \{0,1\}^n \mid a^T x \leq a_0\right\}$ be a knapsack set and $W \in \mathbf{Z}^{d\times n}$ encoded in unary with d fixed.

- The dual problem: $\gamma(w_0) := \min\{a^T x$ subject to $Wx = w_0\}$.
- Dynamic programming / shortest path techniques apply to the dual.
- Choose argmin $\{f(w_0)$ subject to $\gamma(w_0) \leq a_0\}$.

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Theorem (Lee, Onn, W. '07)

For every fixed m and p, there is an algorithm that, given $a_1, \ldots, a_p \in \mathbf{Z}$, $W \in \{a_1, \ldots, a_p\}^{m \times n}$, and a function $f\colon \mathbf{R}^n \to \mathbf{R}$, finds a matroid base B minimizing $f(W\chi^B)$ in time polynomial in n and $\langle a_1, \ldots, a_p \rangle$.

(... can be solved using iterated matroid intersection algorithms.)