New (Practical) Complementary Pivot Algorithms for Market Equilibria

Vijay V. Vazirani

Georgia Tech

Leon Walras, 1874

- Pioneered general equilibrium theory

General Equilibrium Theory
Occupied center stage in Mathematical Economics for over a century

Central tenet

- Markets should operate at equilibrium

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- Markets should operate at equilibrium

  i.e., prices s.t.

  Parity between supply and demand

Do markets even admit equilibrium prices?

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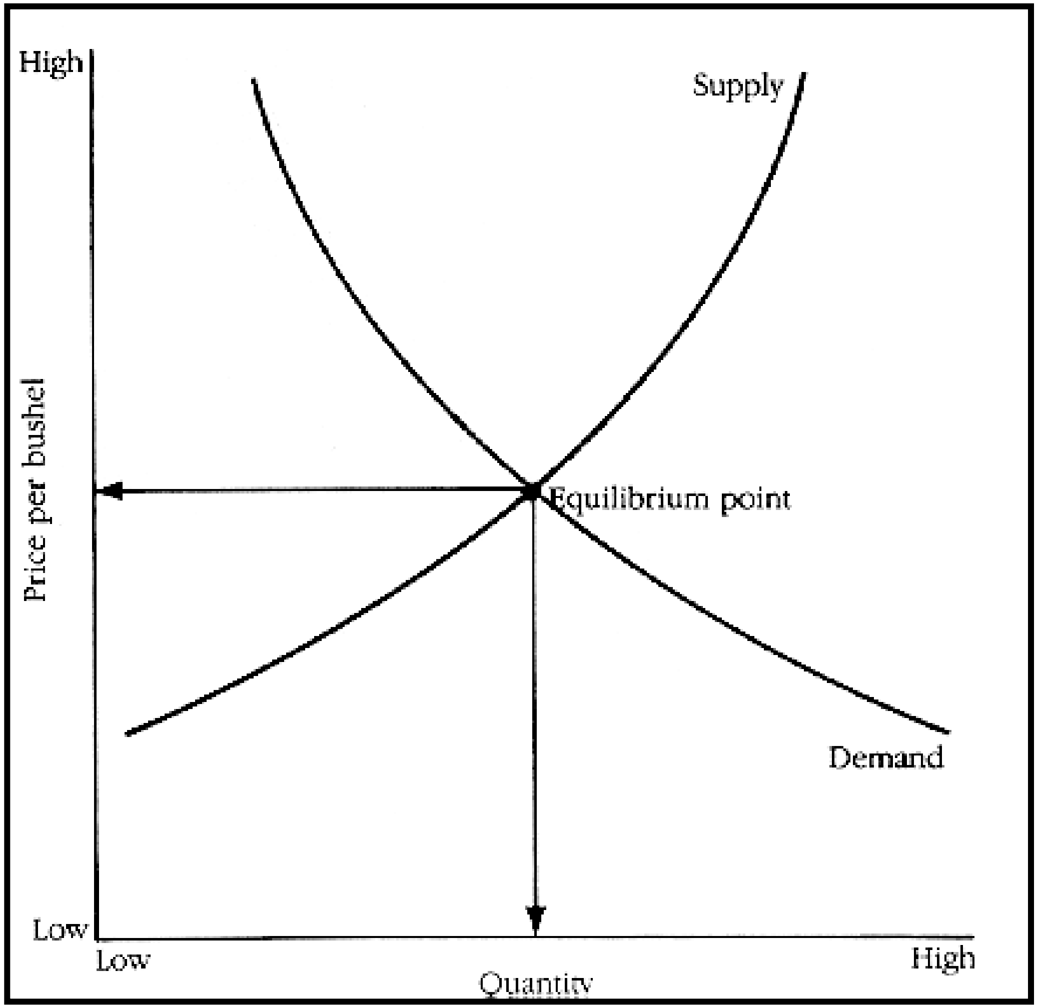
Easy if only one good!

Supply-demand curves



Do markets even admit equilibrium prices?

What if there are multiple goods and multiple buyers with diverse desires and different buying power?

Arrow-Debreu Model

- n agents and g divisible goods.
- Agent i: has initial endowment of goods $\in R_+^g$

  and a concave utility function $u_i : R_+^g \rightarrow R_+$

  (yields convexity!)

Arrow-Debreu Model

- n agents and g divisible goods.
- Agent i: has initial endowment of goods $\in R_+^g$

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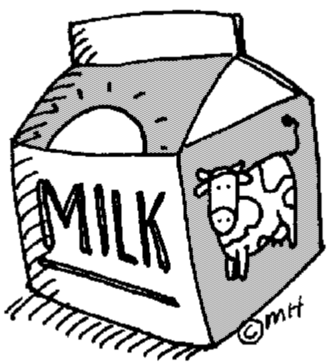
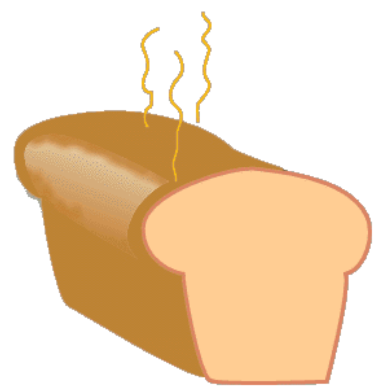
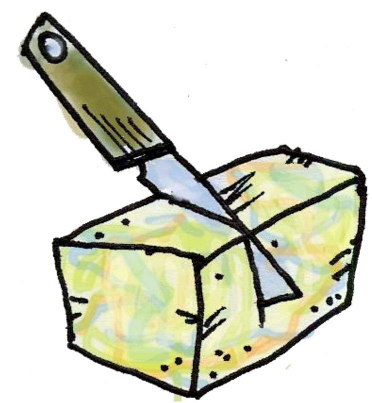
piecewise-linear, concave (PLC)

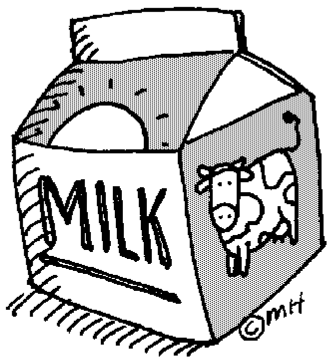
Agent *i* comes with an initial endowment

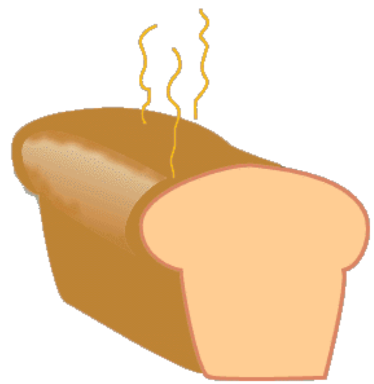


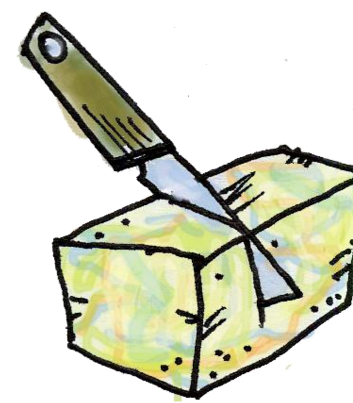




At given prices, agent *i* sells initial endowment

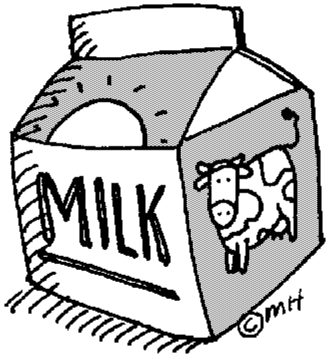


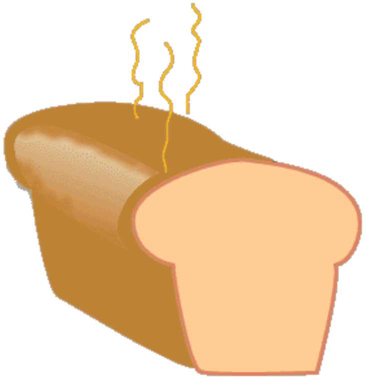


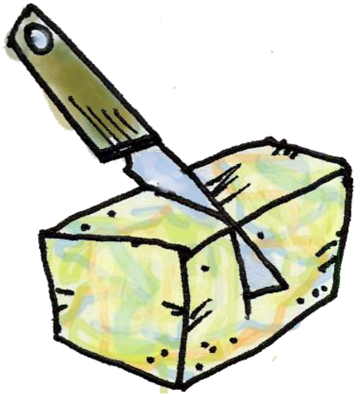


p_1 p_2 p_3

… and buys optimal bundle of goods, i.e, max u_i(bundle)







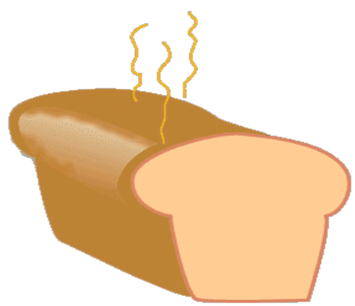
p_1 p_2 p_3

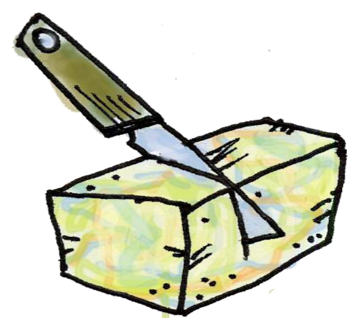
Several agents with own endowments and utility functions.

Currently, no goods in the market.

Agents sell endowments at current prices.



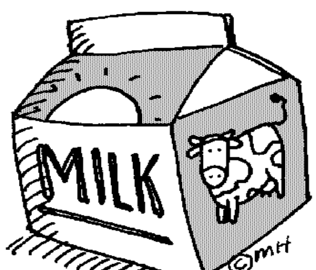



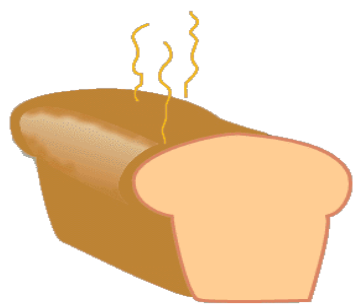


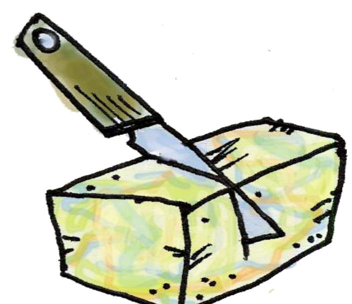
p_1 p_2 p_3

Each agent wants an optimal bundle.







p_1 p_2 p_3

Equilibrium

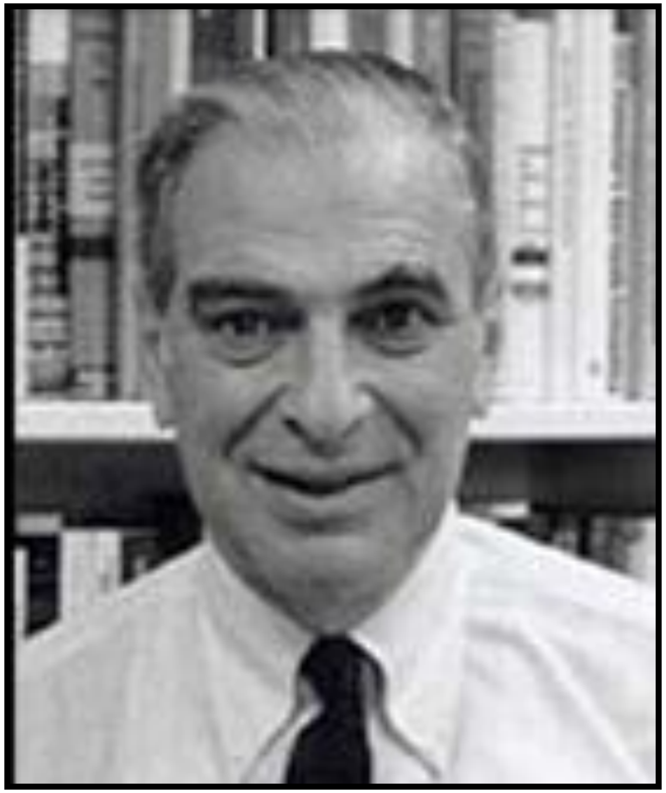
- Prices $\underline{p}$ s.t. market clears,

i.e., there is no deficiency or surplus of any good.

Arrow-Debreu Theorem, 1954

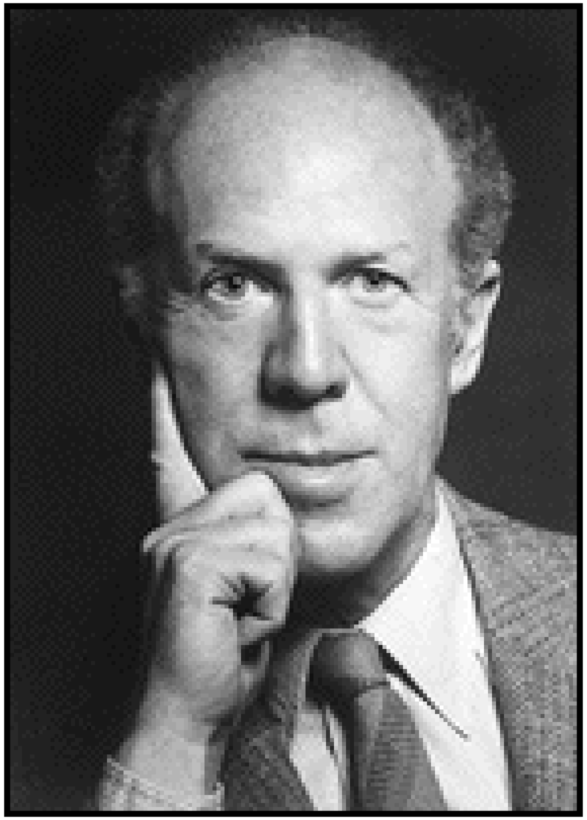
- Celebrated theorem in Mathematical Economics
- Established existence of market equilibrium under very general conditions using a deep theorem from topology - Kakutani fixed point theorem.

Kenneth Arrow



- Nobel Prize, 1972

Gerard Debreu



- Nobel Prize, 1983

Arrow-Debreu Theorem, 1954

- Celebrated theorem in Mathematical Economics
- Established existence of market equilibrium under very general conditions using a theorem from topology - Kakutani fixed point theorem.
- Highly non-constructive!

Inherently algorithmic notion!

- Leon Walras (1774):

  Tatonnement process:

  Price adjustment process to arrive at equilibrium

  - Deficient goods: raise prices
  - Excess goods: lower prices

Leon Walras

- Tatonnement process:

  Price adjustment process to arrive at equilibrium

  - Deficient goods: raise prices
  - Excess goods: lower prices

- Does it converge to equilibrium?

GETTING TO ECONOMIC EQUILIBRIUM: A PROBLEM AND ITS HISTORY

For the third International Workshop on Internet and Network Economics

Kenneth J. Arrow

OUTLINE

I. BEFORE THE FORMULATION OF GENERAL EQUILIBRIUM THEORY
II. PARTIAL EQUILIBRIUM
III. WALRAS, PARETO, AND HICKS
IV. SOCIALISM AND DECENTRALIZATION
V. SAMUELSON AND SUCCESSORS
VI. THE END OF THE PROGRAM

Part VI: THE END OF THE PROGRAM

A. Scarf's example
B. Saari-Simon Theorem: For any dynamic system depending on first-order information (z) only, there is a set of excess demand functions for which stability fails. (In fact, theorem is stronger).
C. Uzawa: Existence of general equilibrium is equivalent to fixed-point theorem
D. Assumptions on individual demand functions do not constrain aggregate demand function (Sonnenschein, Debreu, Mantel)

Centralized algorithms for equilibria

- Scarf, Smale, … , 1970s: Nice approaches!

Centralized algorithms for equilibria

- Scarf, Smale, … , 1970s: Nice approaches!

  (slow and suffer from numerical instability)


HANDBOOKS
IN ECONOMICS
Peter B. Dixon
Dale Jorgenson
Handbook of Computable General Equilibrium Modeling
VOLUME 1B
NORTH HOLLAND

Theoretical Computer Science

- Primal-dual paradigm
- Convex programs
- Complementary pivot algorithms

(Complementary) Pivot Algorithms

- Dantzig, 1947: Simplex algorithm for LP
- Lemke-Howson, 1964: 2-Nash Equilibrium
- Eaves, 1975: Equilibrium for Arrow-Debreu markets under linear utilities

$$f(\underline{x}) = \sum_j c_j x_j$$

(Complementary) Pivot Algorithms

- Very fast in practice (even though exponential time in worst case).
- Work on rational numbers with bounded denominators, hence no instability issues.
- Reveal deep structural properties.

(Complementary) Pivot Algorithms

- Eaves, 1975: Equilibrium for linear
  Arrow-Debreu markets
  (based on Lemke's algorithm)

- Until very recently, no extension to more general utility functions!

(Complementary) Pivot Algorithms

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  Arrow-Debreu markets

- Until very recently, no extension to more general utility functions!

Why?

Separable, piecewise-linear concave utility functions

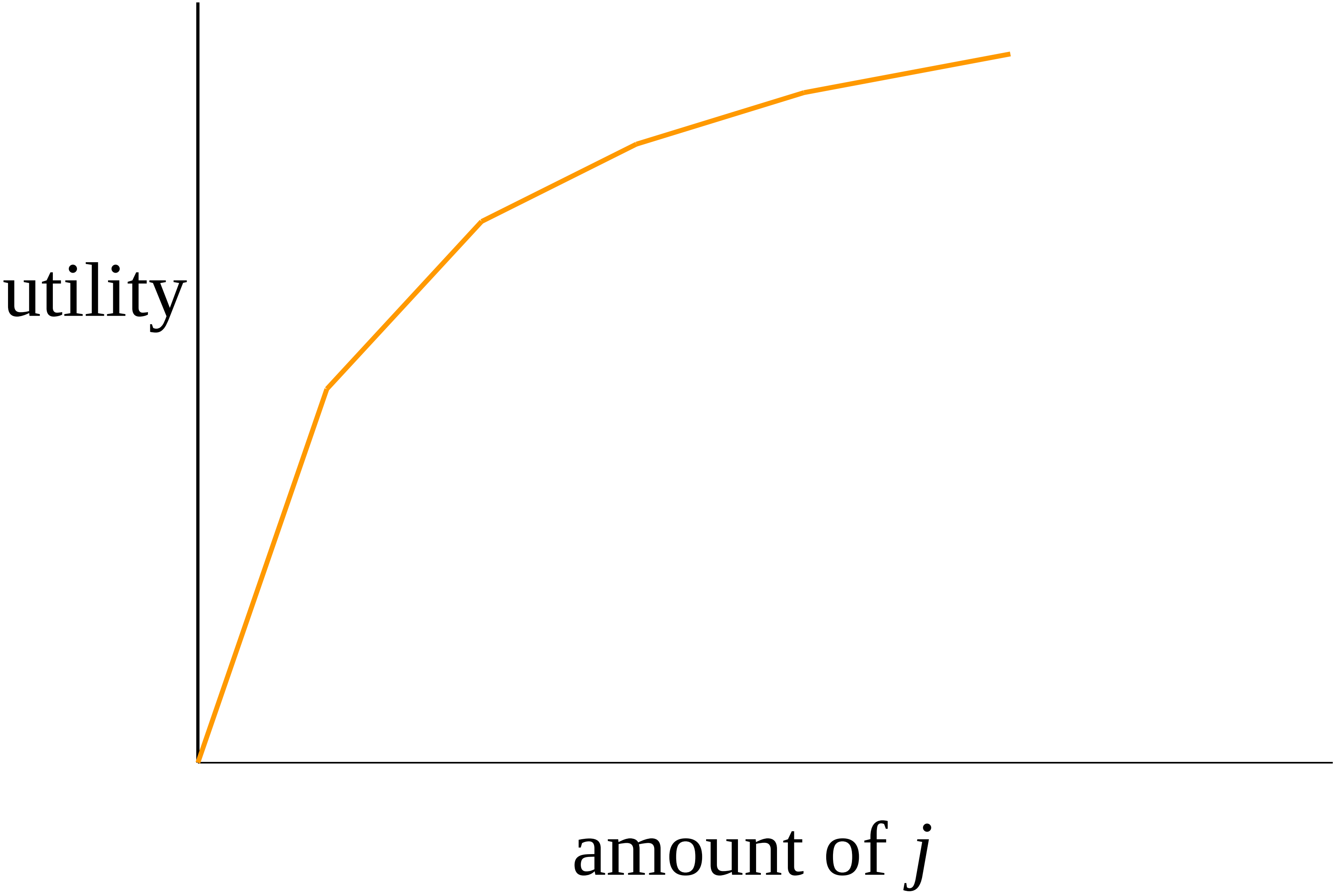
Separable utility function

For a single buyer :

Utility from good j, $f_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$

Total utility from bundle, $f(\underline{x}) = \sum_j f_j(x_j)$

f_j : piecewise-linear, concave



Arrow-Debreu market under separable, piecewise-linear concave (SPLC) utilities

- Can Eaves' algorithm be extended to this case?

- Eaves, 1975 Technical Report:

Also under study are extensions of the overall method to include piecewise-linear utilities, production, etc., if successful, this avenue could prove important in real economic modeling.

- Eaves, 1976 Journal Paper:

... Now suppose each trader has a piecewise-linear, concave utility function. Does there exist a rational equilibrium? Andreu Mas-Colell generated a negative example, using Leontief utilities. Consequently, one can conclude that Lemke's algorithm cannot be used to solve this class of exchange problems.

Leontief utility

$$u(\underline{x}) = \min\left(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \;\ldots\;, \frac{x_n}{a_n}\right)$$

Leontief utility: is non-separable!

- Utility = min{#bread, 2 #butter}
- Only bread or only butter gives 0 utility!

Rationality for SPLC Utilities

- Devanur & Kannan, 2007,
  V. & Yannakakis, 2007:

- If all parameters are rational numbers,
  there is a rational equilibrium.

Theorem (Garg, Mehta, Sohoni & V., 2012):

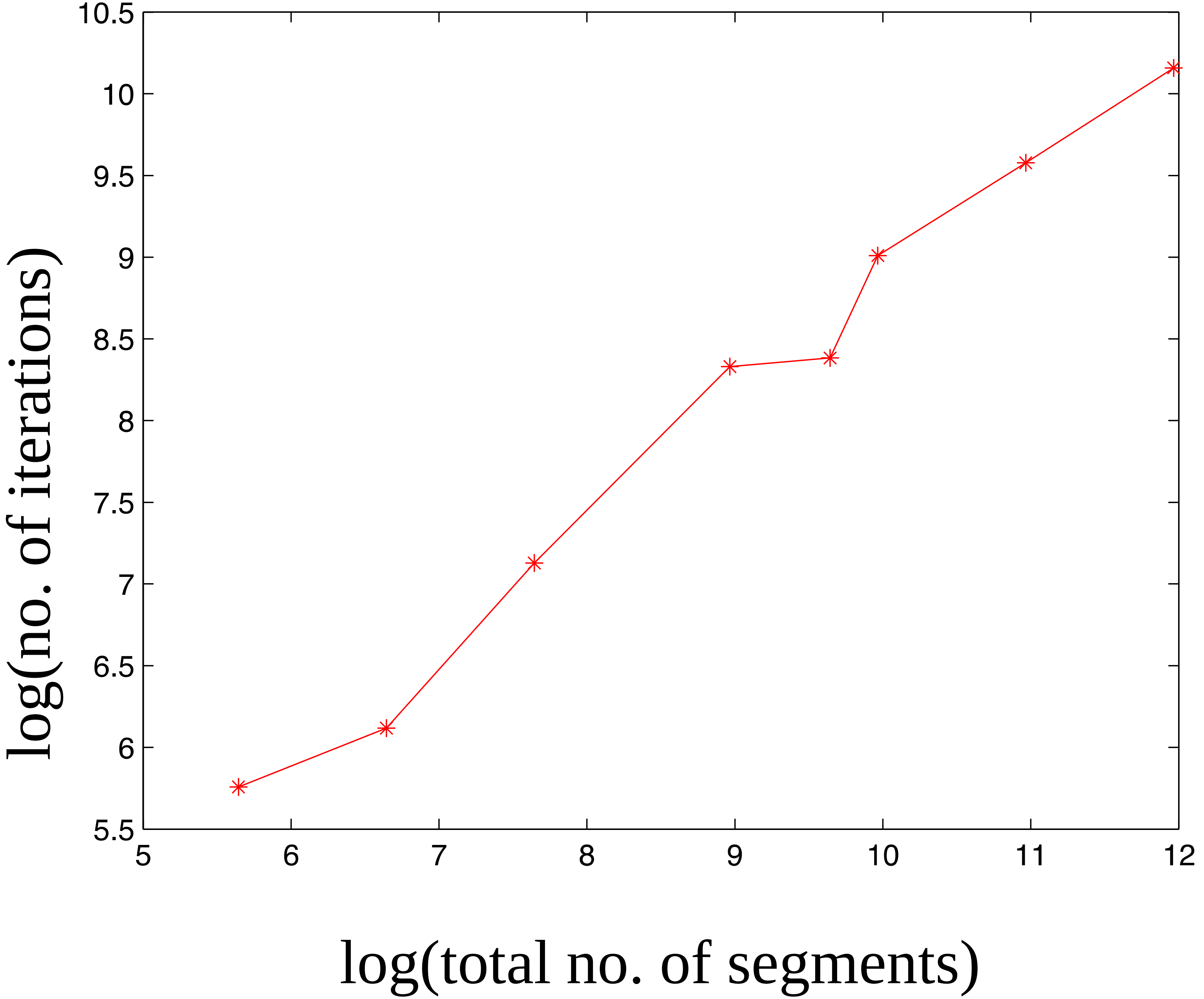
Complementary pivot algorithm for Arrow-Debreu markets under SPLC utility functions.

(based on Lemke's algorithm)

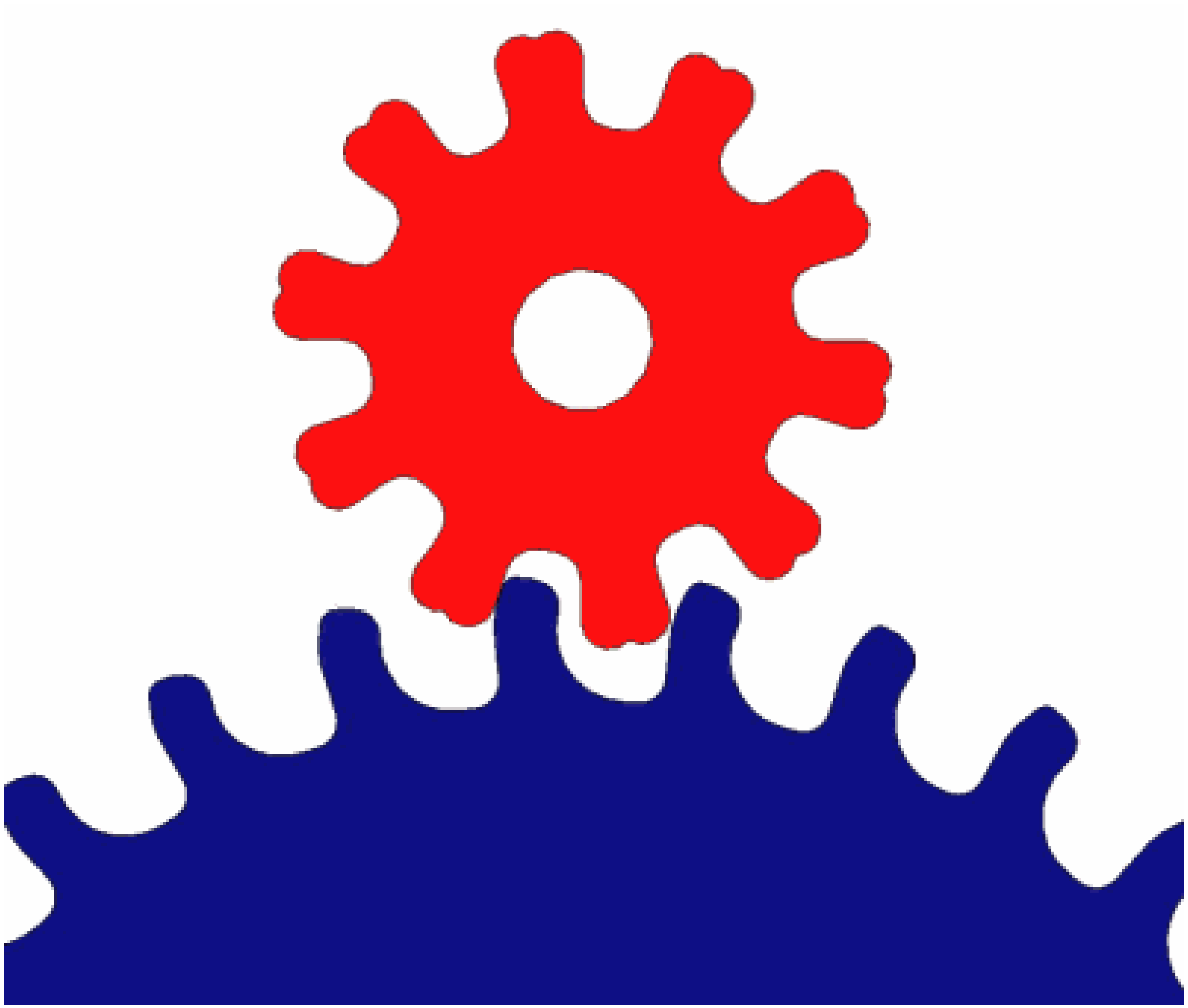
Experimental Results

- Inputs are drawn uniformly at random.

|A|x|G|x#Seg	#Instances	Min Iters	Avg Iters	Max Iters
10 x 5 x 2	1000	55	69.5	91
10 x 5 x 5	1000	130	154.3	197
10 x 10 x 5	100	254	321.9	401
10 x 10 x 10	50	473	515.8	569
15 x 15 x 10	40	775	890.5	986
15 x 15 x 15	5	1203	1261.3	1382
20 x 20 x 5	10	719	764	853
20 x 20 x 10	5	1093	1143.8	1233


10.5
10
9.5
9
8.5
8
7.5
7
6.5
6
5.5
5
6
7
8
9
10
11
12
log(no. of iterations)
log(total no. of segments)

Linear Complementarity & Lemke's Algorithm



Linear complementarity problem

- Generalizes LP

$$
\begin{aligned}
\min \quad & \underline{c}^T \underline{x} \\
s.t. \quad & A\underline{x} \geq \underline{b} \\
& \underline{x} \geq 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\max \quad & \underline{b}^T \underline{z} \\
s.t. \quad & A^T \underline{z} \leq \underline{c} \\
& \underline{z} \geq 0
\end{aligned}
$$

$$\min \ \underline{c}^T \underline{x} \qquad\qquad \max \ \underline{b}^T \underline{z}$$

$$s.t. \ A\underline{x} \geq \underline{b} \qquad\qquad s.t. \ A^T \underline{z} \leq c$$

$$\underline{x} \geq 0 \qquad\qquad \underline{z} \geq 0$$

Complementary Slackness:

Let $\underline{x}$, $\underline{z}$ both be feasible.

Then both are optimal iff

$$\forall i: \ z_i = 0 \ \text{ or } \ \left(A\underline{x}\right)_i = b_i$$

$$\forall j: \ x_j = 0 \ \text{ or } \ \left(A^T \underline{z}\right)_j = c_j$$

$\underline{x}$, $\underline{z}$ are both optimal iff

$$A\underline{x} \geq \underline{b} \qquad A^T\underline{z} \leq \underline{c}$$

$$\underline{x} \geq 0 \qquad \underline{z} \geq 0$$

$$\underline{z}^T\left(A\underline{x} - \underline{b}\right) = 0 \qquad \underline{x}^T\left(\underline{c} - A^T\underline{z}\right) = 0$$

Let

square matrix

$$\underline{M} = \begin{pmatrix} -A & 0 \\ 0 & A^T \end{pmatrix} \quad \underline{y} = \begin{pmatrix} \underline{x} \\ \underline{z} \end{pmatrix} \quad \underline{q} = \begin{pmatrix} -\underline{b} \\ \underline{c} \end{pmatrix}$$

Then $\underline{y}$ gives optimal solutions iff

$$M\,\underline{y} \le \underline{q}$$

$$\underline{y} \ge 0$$

$$\underline{y}.\left(\underline{q} - M\,\underline{y}\right) = 0$$

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Linear Complementarity Problem

Given $n \times n$ matrix M and vector $\underline{q}$ find $\underline{y}$ s.t.

$$M\underline{y} \le \underline{q}$$

$$\underline{y} \ge 0$$

$$\underline{y} . (\underline{q} - M\underline{y}) = 0$$

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quadratic

Linear Complementarity Problem

Given $n \times n$ matrix M and vector $\underline{q}$ find $\underline{y}$ s.t.

$$M\underline{y} \le \underline{q}$$

$$\underline{y} \ge 0$$

$$\underline{y}.(\underline{q} - M\underline{y}) = 0 \quad \left(\text{Clearly, } \underline{q} - M\underline{y} \ge 0\right)$$

i.e., for each i:

$y_i = 0$ or inequality i is satisfied with equality.

Examples of linear complementarity

- LP: complementary slackness

- 2-Nash: For row player,

either Pr[row *i*] = 0 or row *i* is a best response.

Nonlinear complementarity

- Plays a key role in KKT conditions for convex programs

Linear Complementarity Problem

Given $n \times n$ matrix M and vector $\underline{q}$ find $\underline{y}$ s.t.

$$M\underline{y} \le \underline{q}$$

$$\underline{y} \ge 0$$

$$\underline{y} \cdot (\underline{q} - M\underline{y}) = 0$$

i.e., for each i:

$y_i = 0$ or inequality i is satisfied with equality.

Linear Complementarity Problem

Given $n \times n$ matrix M and vector $\underline{q}$ find $\underline{y}$ s.t.

$$M\,\underline{y} \le \underline{q}$$

$$\underline{y} \ge 0$$

$$\underline{y}.(\underline{q} - M\,\underline{y}) = 0$$

.

Introduce slack variables $\underline{v}$

$$M\,\underline{y} + \underline{v} = \underline{q}$$

$$\underline{y} \ge 0, \qquad \underline{v} \ge 0 \qquad \text{(Since } \underline{q} - M\,\underline{y} \ge 0\text{)}$$

$$\underline{y}.\,\underline{v} = 0$$

i.e., for each i: $y_i = 0$ or $v_i = 0$.

$$M\underline{y} + \underline{v} = \underline{q}$$

$$\underline{y} \geq 0$$

$$\underline{v} \geq 0$$

$$\underline{y} \cdot \underline{v} = 0$$

Assume polyhedron in $\mathbb{R}^{2n}$ defined by red constraints is non-degenerate.

Solution to LCP satisfies $2n$ equalities

$\Rightarrow$ is a vertex of the polyhedron.

Possible scheme

- Find one vertex of polyhedron
  and walk along 1-skeleton, via pivoting,
  to a solution.

Possible scheme

- Find one vertex of polyhedron
  and walk along 1-skeleton, via pivoting,
  to a solution.

- But in which direction is the solution?

Lemke's idea

Add a new dimension:

$$M\,\underline{y} + \underline{v} - z\underline{1} = \underline{q}$$

$$\underline{y} \geq 0$$

$$\underline{v} \geq 0$$

$$z \geq 0$$

$$\underline{y} \cdot \underline{v} = 0$$

Lemke's idea

Add a new dimension:

$$M\underline{y} + \underline{v} - z\underline{1} = \underline{q}$$

$$\underline{y} \geq 0$$

$$\underline{v} \geq 0$$

$$z \geq 0$$

$$\underline{y}.\ \underline{v} = 0$$

Note: Easy to get a solution to augmented LCP:

Pick $\underline{y} = 0$, z large and $\underline{v} = \underline{q} + z\underline{1}$. Then $\underline{v} \geq 0$.

Lemke's idea

Add a new dimension:

$$M\underline{y} + \underline{v} - z\underline{1} = \underline{q}$$

$$\underline{y} \geq 0$$

$$\underline{v} \geq 0$$

$$z \geq 0$$

$$\underline{y} \cdot \underline{v} = 0$$

Want: solution of augmented LCP with $z = 0$.

Will be solution of original LCP!

- *S*: set of solutions to augmented LCP,
  each satisfies $2n$ equalities.

- Polyhedron is in $2n+1$ space.

- Hence, *S* is a subset of 1-skeleton, i.e.,
  consist of edges and vertices.

- Every solution is fully labeled, i.e.,

$$\forall i : y_i = 0 \text{ or } v_i = 0$$

Vertices of polyhedron lying in *S*

- Two possibilities:

1). Has a double label, i.e.,

$$\$ i : y_i = 0 \quad \text{and} \quad v_i = 0$$

- Only 2 ways of relaxing double label, hence this vertex has exactly 2 edges of *S* incident.

Vertices of polyhedron lying in S

- Two possibilities:

2). Has $z = 0$

- Only 1 way of relaxing $z = 0$, hence this vertex has exactly 1 edge of S incident.

Vertices of polyhedron lying in S

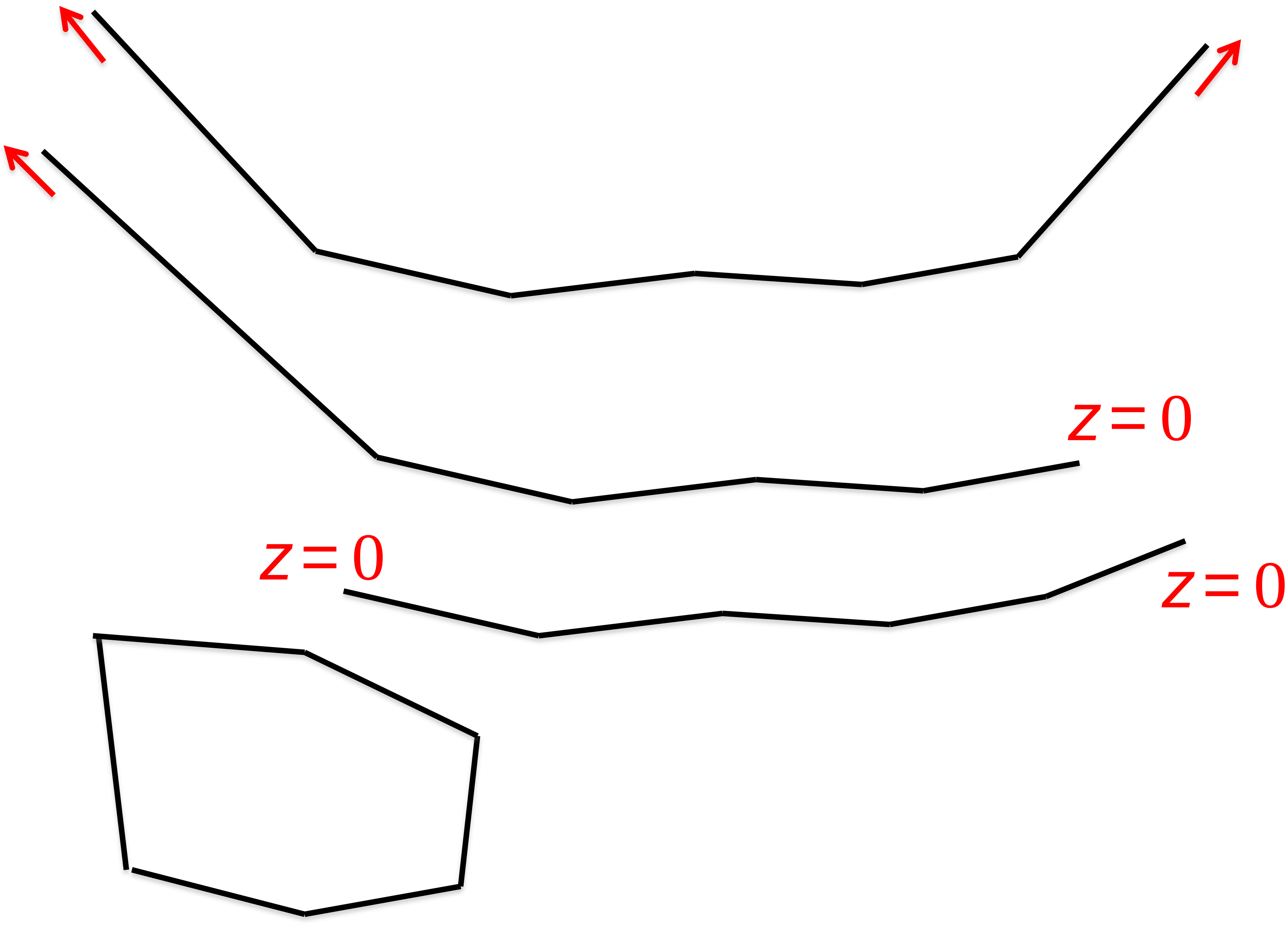
- Two possibilities:

2). Has $z = 0$

- Only 1 way of relaxing $z = 0$, hence this vertex has exactly 1 edge of S incident.

- This is a solution to original LCP!

Hence S consists of paths and cycles!


z = 0
z = 0
z = 0

- ray: unbounded edge of *S*.
- principal ray: each point has $\underline{y} = 0$.
- secondary ray: rest of the rays.

Lemke's idea

Add a new dimension:

$$M\underline{y} + \underline{v} - z\underline{1} = \underline{q}$$

$$\underline{y}^{3}\ 0$$

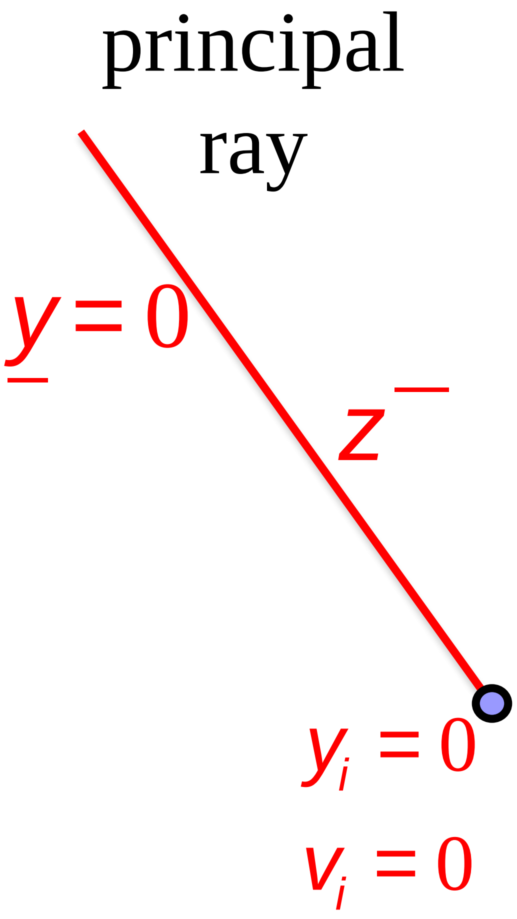
$$\underline{v}^{3}\ 0$$

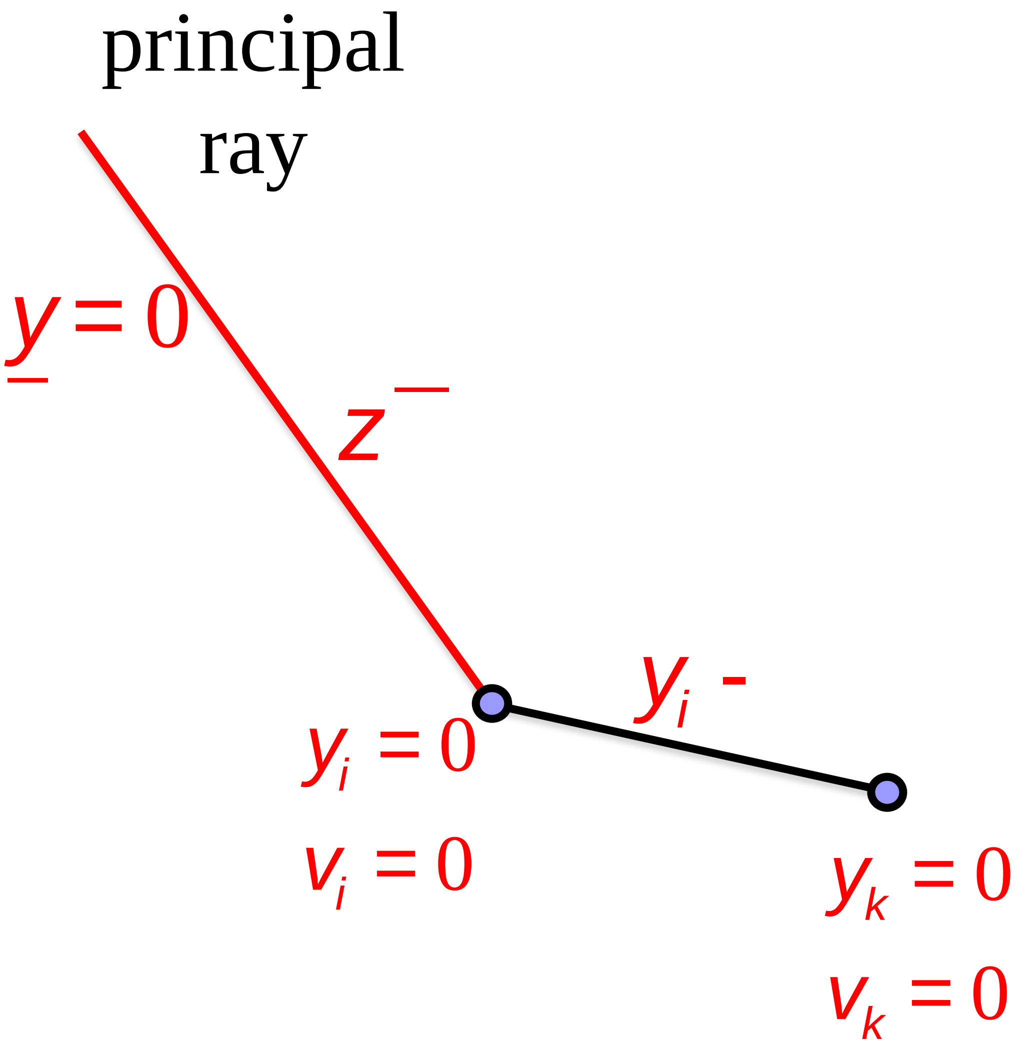
$$z^{3}\ 0$$

$$\underline{y}.\ \underline{v} = 0$$

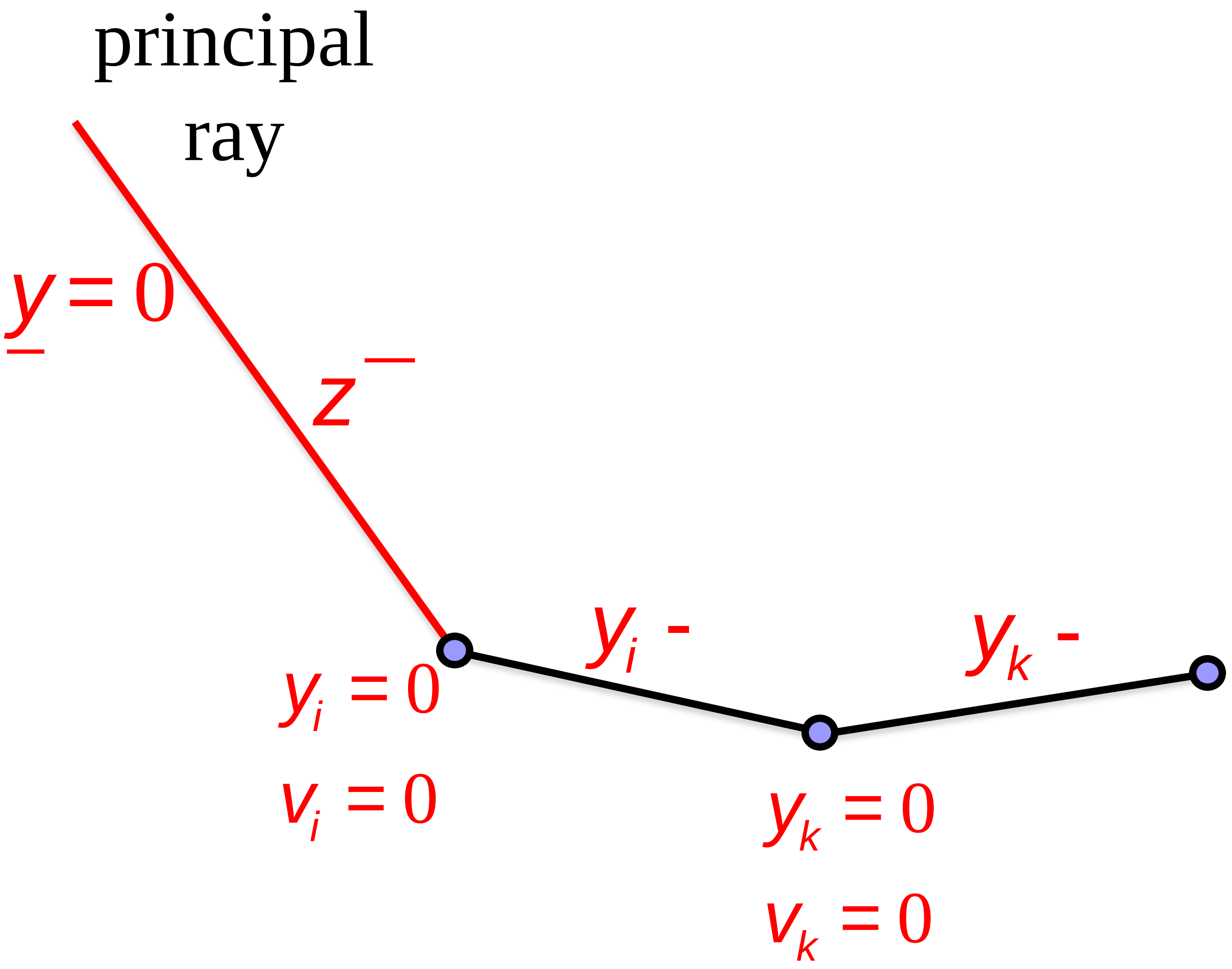
Note: Easy to get a solution to augmented LCP:

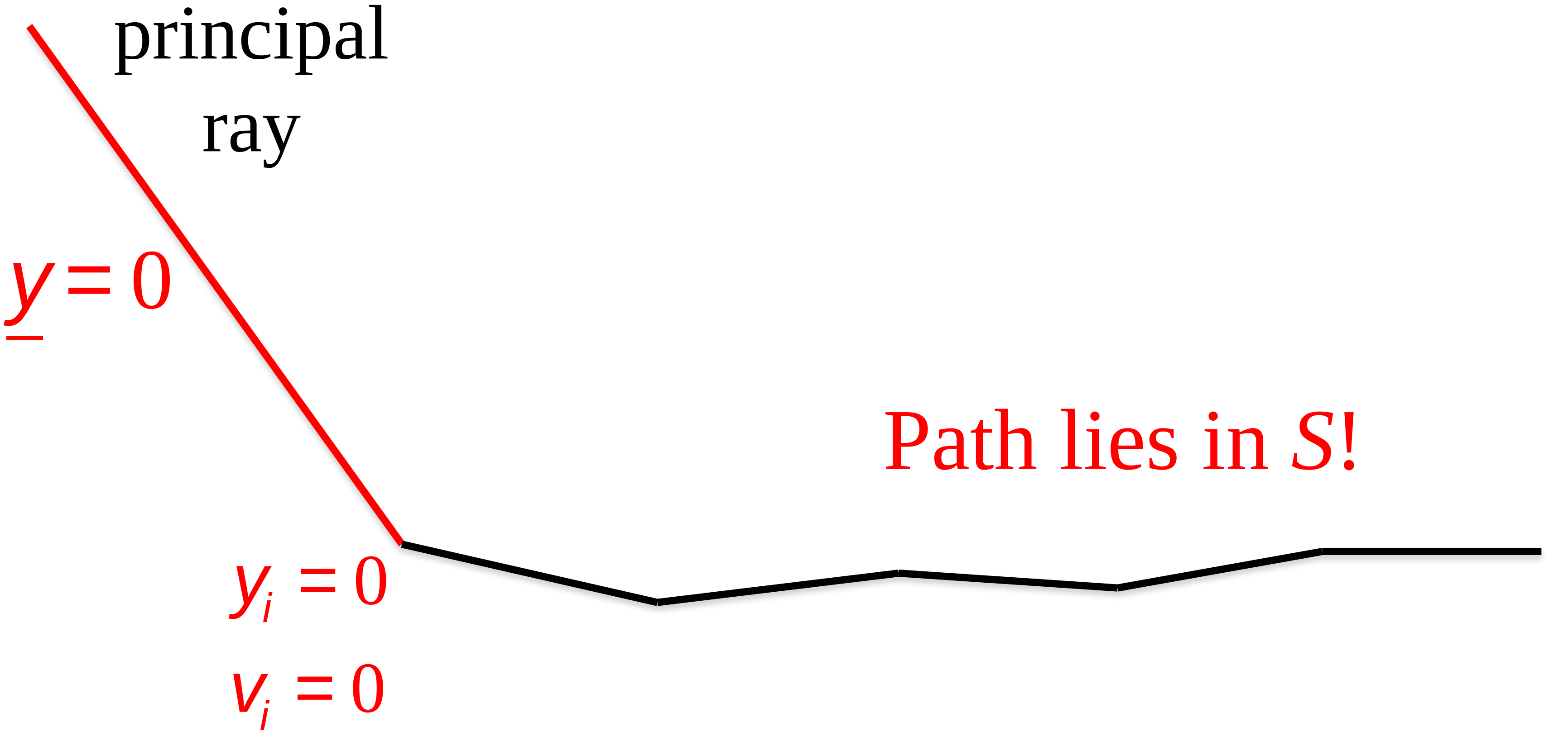
Pick $\underline{y} = 0$, z large and $\underline{v} = \underline{q} + z\underline{1}$. Then $\underline{v}^{3}\ 0$.

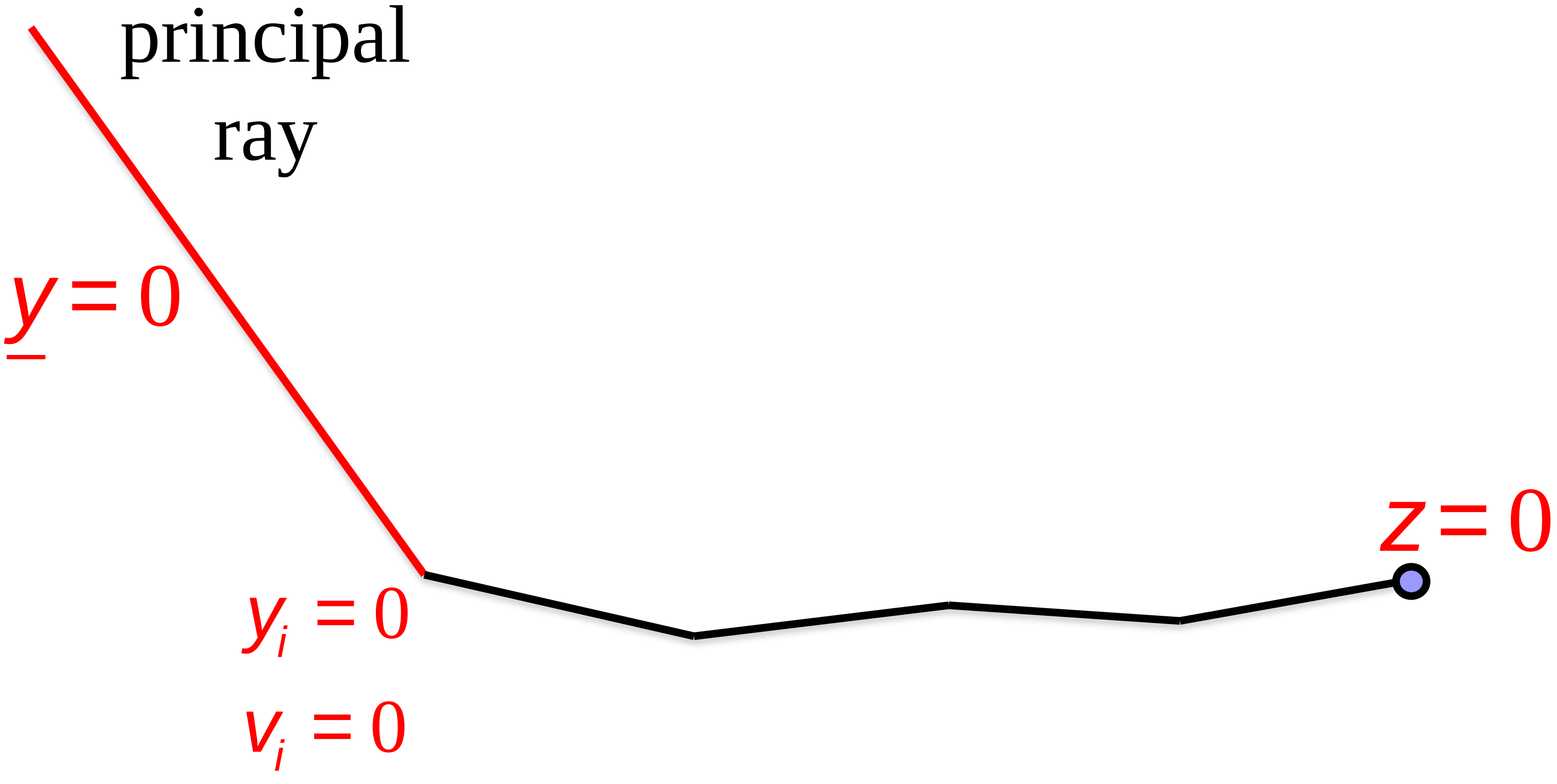

principal
ray
y = 0
z
y_i = 0
v_i = 0

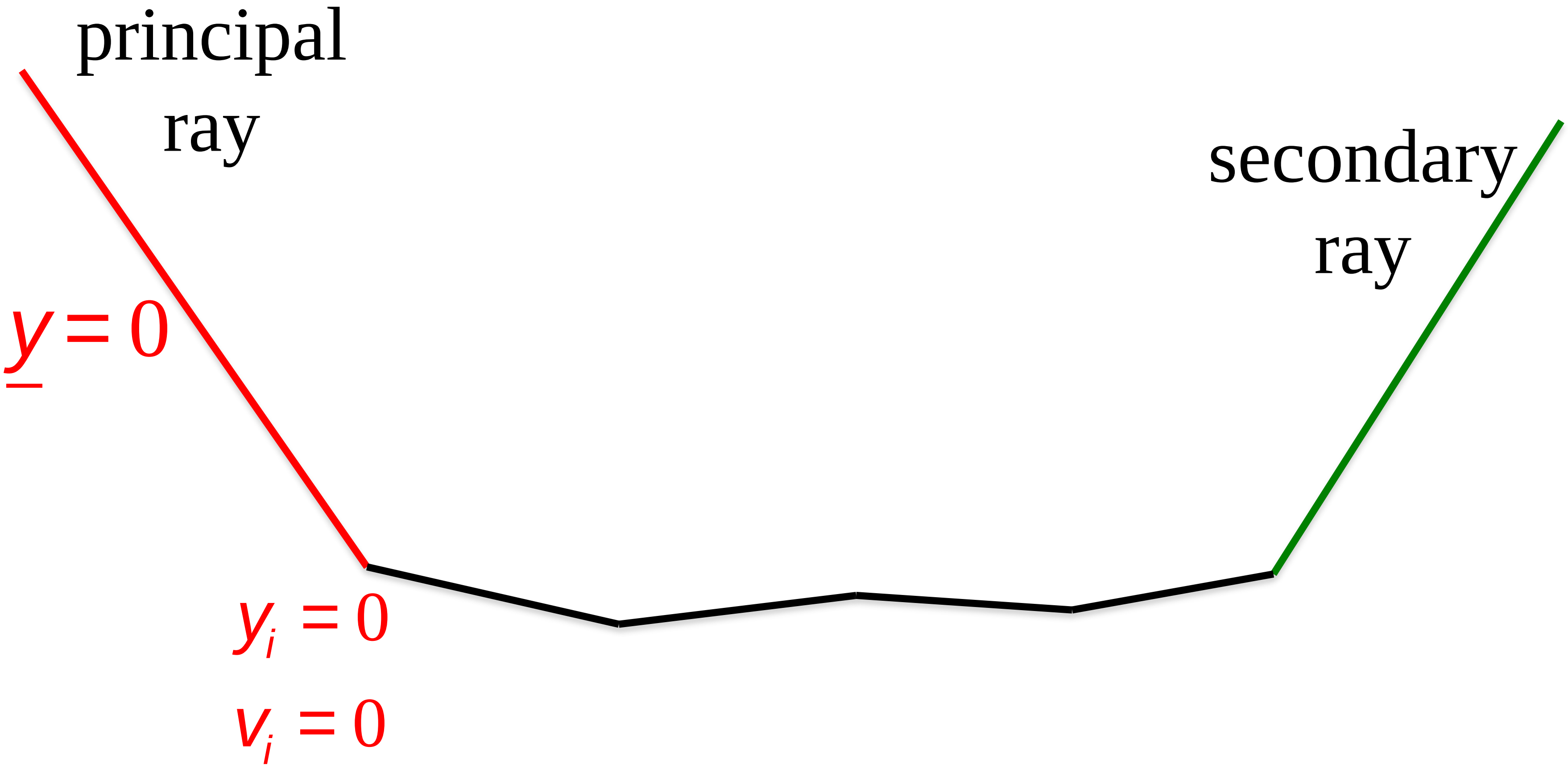


principal ray
$\underline{y} = 0$
$\overline{z}$
y_i -
$y_i = 0$
$v_i = 0$
$y_k = 0$
$v_k = 0$


principal ray
$\underline{y} = 0$
$\underline{z}$
y_i -
y_k -
$y_i = 0$
$v_i = 0$
$y_k = 0$
$v_k = 0$


principal
ray
$\underline{y} = 0$
Path lies in S!
$y_i = 0$
$v_i = 0$


principal
ray
$\underline{y} = 0$
$y_i = 0$
$v_i = 0$
$z = 0$


principal
ray
$\underline{y} = 0$
$y_i = 0$
$v_i = 0$
secondary
ray

Problem with Lemke's algorithm

- No recourse if path starting with primary ray ends in a secondary ray!

Problem with Lemke's algorithm

- No recourse if path starting with primary ray ends in a secondary ray!
- We show that for each of our LCPs, associated polyhedron has no secondary rays!

Dramatic change!

Polyhedron of

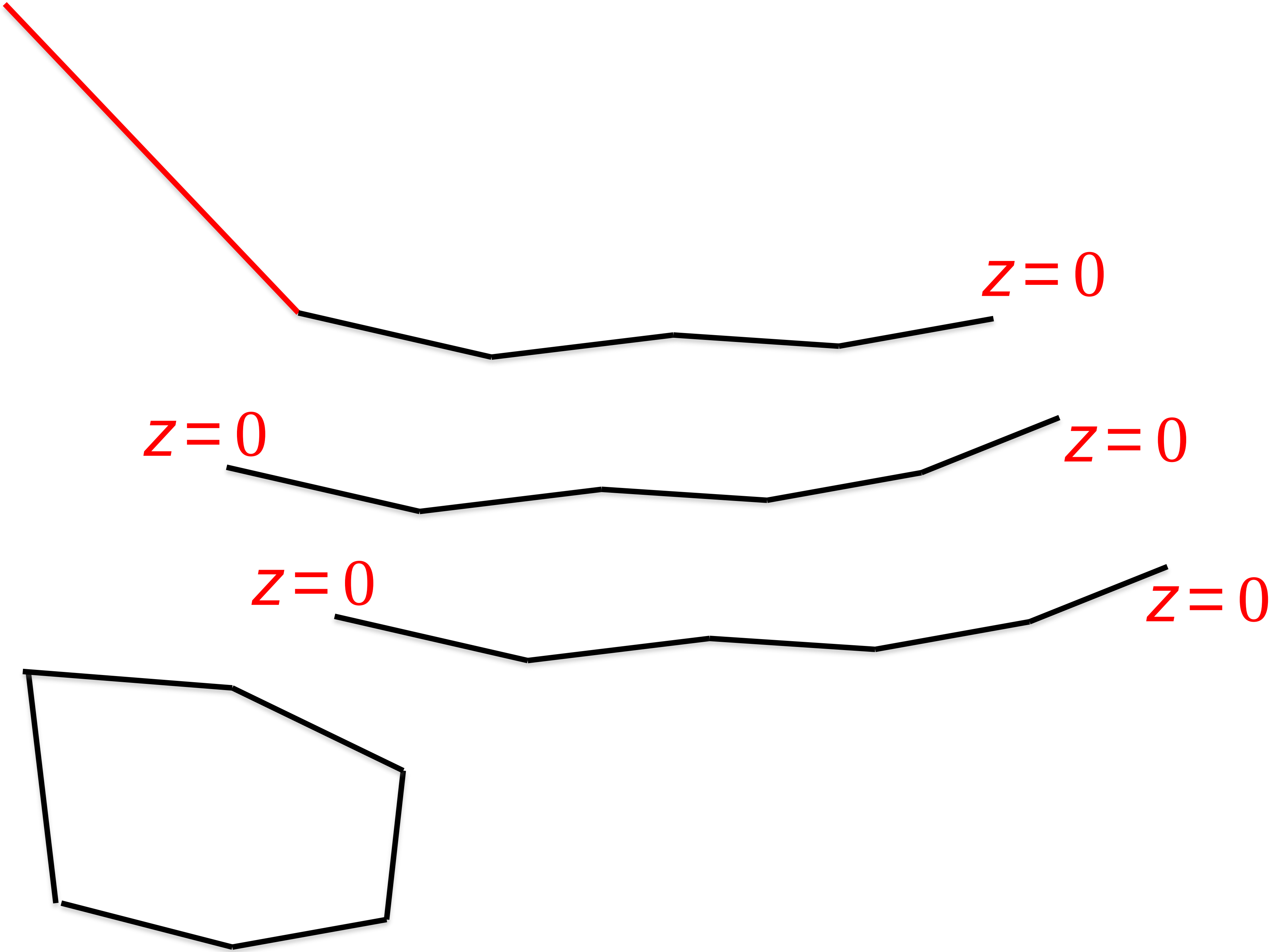
- original LPC: no clue where solution is.
- augmented LCP: know a path
  leading to solution!

Theorem (Garg, Mehta, Sohoni & V., 2012):

1). Derive LCP whose solutions correspond to equilibria.

2). Polyhedron of LCP has no secondary rays.

Corollary: The number of equilibria is odd, up to scaling.



z = 0
z = 0
z = 0
z = 0
z = 0

Theorem (Garg, Mehta, Sohoni & V., 2012):

1). Derive LCP whose solutions correspond to equilibria.

2). Polyhedron of LCP has no secondary rays.

3). If no. of goods or agents is a constant, polyn. vertices of polyhedron are solutions

=> strongly polynomial algorithm

Derive LCP (assume linear utilities)

- Market clearing
  - Every good fully sold
  - Every agent spends all his money
- Optimal bundles
  - Every agent gets a utility maximizing bundle

Model

Utility of agent i: $\sum_j u_{ij} x_{ij}$

Initial endowment of agent i: $w_{ij}, \; j \in G$

W.l.o.g. assume 1 unit of each good in the market.

Variables

p_j : price of good j

q_{ij} : amount of money spent by i on j

Guaranteeing optimal bundles

- Agent i spends only on $S_i = \arg\max_j \left\{ \frac{u_{ij}}{p_j} \right\}$

- bang-per-buck of i $= \max_j \left\{ \frac{u_{ij}}{p_j} \right\}$

  $\left(= \frac{1}{z_i} \text{ at equilibrium}\right)$

Optimal bundles, guaranteed by:

$$\forall j: \quad \frac{u_{ij}}{p_j} \le \frac{1}{\lambda_i}$$

$$q_{ij} > 0 \Rightarrow \frac{u_{ij}}{p_j} = \frac{1}{\lambda_i}$$

Optimal bundles, guaranteed by:

$$" j : \qquad \frac{u_{ij}}{p_j} \; £ \; \frac{1}{l_i}$$

$$q_{ij} > 0 \; Þ \; \frac{u_{ij}}{p_j} = \frac{1}{l_i}$$

$$q_{ij} > 0 \;\; \text{or} \;\; \frac{u_{ij}}{p_j} = \frac{1}{l_i}$$

Optimal bundles, via complementarity

$$" i : " j : \quad u_{ij} / _{i} \text{ £ } p_{j}$$

$$q_{ij} (u_{ij} / _{i} - p_{j}) = 0$$

LCP for linear utilities

$$\forall j: \sum_i q_{ij} \le p_j \qquad \text{comp} \quad p_j$$

$$\forall i: \sum_j w_{ij} p_j \le \sum_j q_{ij} \qquad \text{comp} \quad \lambda_i$$

$$\forall i, j: u_{ij} \lambda_i \le p_j \qquad \text{comp} \quad q_{ij}$$

& non-negativity for $p_j, \; q_{ij}, \; \lambda_i$