Smoothed Analysis of Algorithms
Part II: Binary and Multiobjective Optimization

Heiko Röglin
Department of Computer Science
universitätbonn

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Outline

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1. **Binary Optimization Problems**
   When does a binary optimization problem have polynomial smoothed complexity?
2. **Multiobjective Optimization**
   How many Pareto-optimal solutions do usually exist?
3. **Conclusions**

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Model

Linear Binary Optimization Problem

- set of feasible solutions $\mathcal{S} \subseteq \{0,1\}^n$
  solution $x = (x_1, \ldots, x_n) \in \mathcal{S}$ consists of n binary variables
- linear objective function
  $\max c^T x = c_1 x_1 + \cdots + c_n x_n$

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$\mathcal{S}$ can encode arbitrary combinatorial structure, e.g., for a given graph, all paths from s to t, all Hamiltonian cycles, all spanning trees, ...

- Knapsack Problem: variable $x_i \in \{0,1\}$ for each item i
  $\mathcal{S} = \{x \mid w_1 x_1 + \cdots + w_n x_n \leq t\}$

- TSP: variable $x_e \in \{0,1\}$ for each $e \in E$
  $\mathcal{S} = \{x \mid x \text{ encodes Hamiltonian cycle}\}$

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Smoothed Analysis:
Adversary chooses $\mathcal{S}$ and a probability density
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Every c_i is **drawn independently** according to f_i.

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Remarks:

- ϕ large $\approx$ worst case
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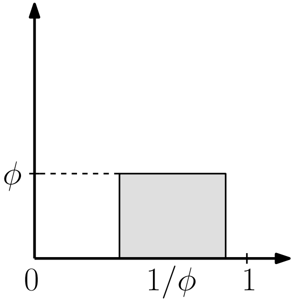

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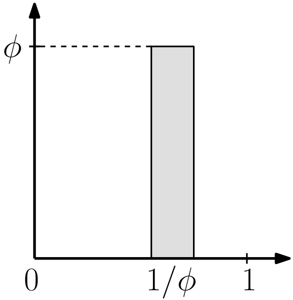

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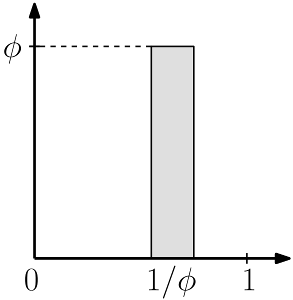

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- $\mathcal{S}$ is not perturbed!

Main Result

Theorem [Beier, Vöcking (STOC 2004)]

linear binary opt. problem has polynomial smoothed complexity

$$\Longleftrightarrow$$

pseudo-polynomial time $\mathrm{poly}(n, \max\{|c_i|\})$ in the worst case

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  $\Rightarrow$ polynomial smoothed complexity

- **TSP**: strongly NP-hard
  (even if all edge lengths are 1 or 2)
  $\Rightarrow$ no polynomial smoothed complexity

Polynomial Smoothed Complexity

- A = algorithm
- $\mathcal{I}_n$ = set of inputs of length n
- $\mathrm{per}_\phi(I)$ = perturbation of instance I
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Definition

Algorithm A has polynomial smoothed complexity if there exist $\alpha > 0$ and $\beta > 0$ with

$$\max_{I\in\mathcal{I}_n} \mathbf{E}\left[T_A(\mathrm{per}_\phi(I))^\alpha\right] \leq \beta n \phi.$$

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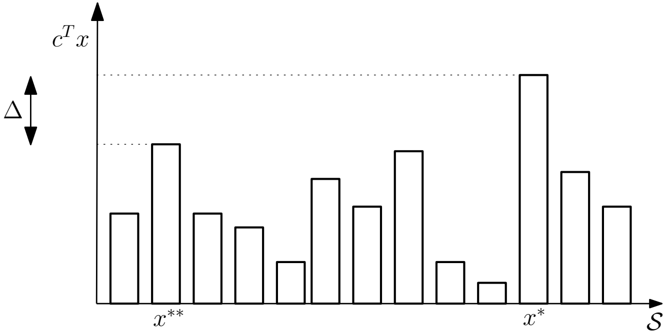

$$\Delta = c^T x^* - c^T x^{**}$$

$$c_1 = 0.8961235$$
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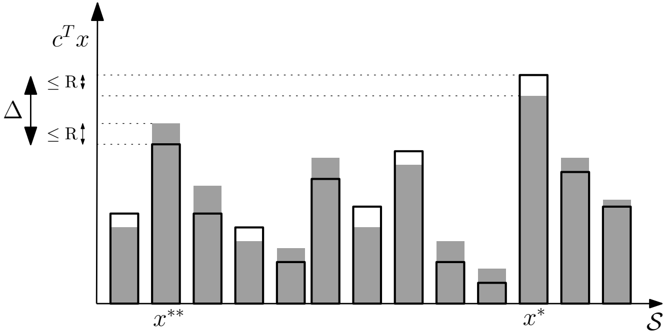

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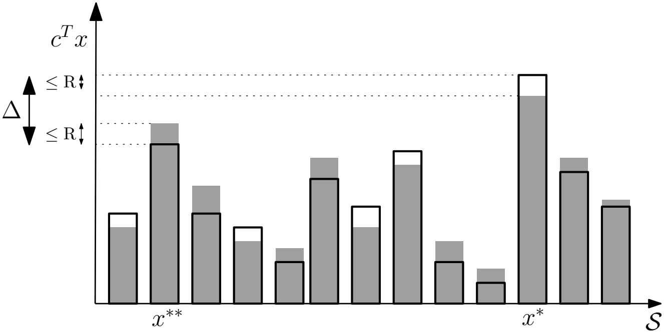

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- Rounding after the b-th bit $\Rightarrow |c_i - [c_i]| \leq 2^{-b}$
  $\Rightarrow \forall x \in \mathcal{S} : |c^T x - [c]^T x| \leq n2^{-b} = R$

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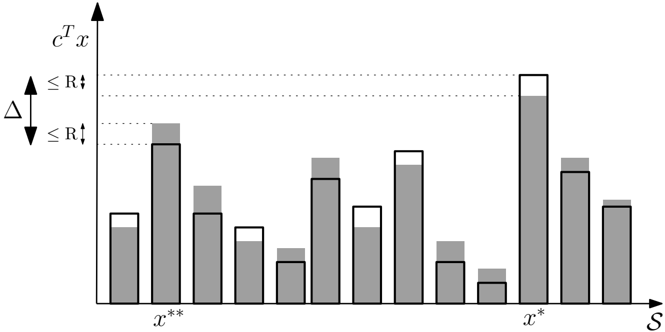

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- $\Delta > 2R \Rightarrow$ rounding does not change optimal solution

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For every $p \in (0, 1]$ and $b \geq \log\left(\frac{n^2\phi}{p}\right) + 2$,

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- Round coefficients after a logarithmic number of bits and call pseudo-polynomial algorithm.
- If necessary, increase precision and repeat.

Proof of Isolation Lemma

Lemma

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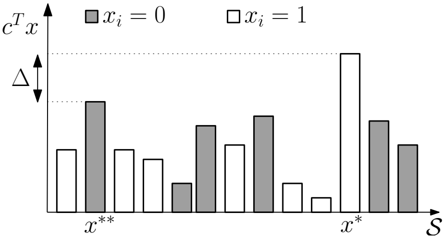

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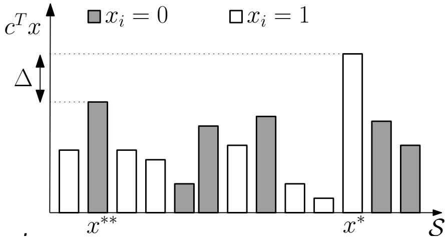

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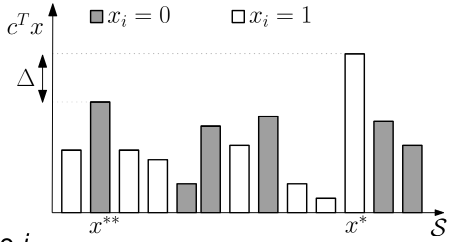

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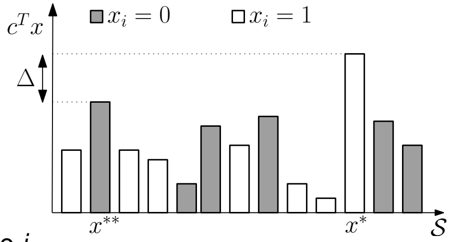

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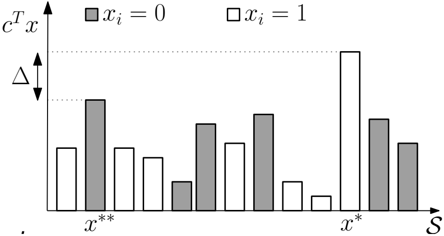

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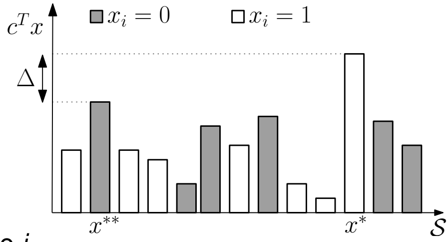

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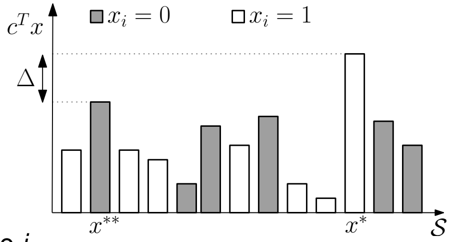

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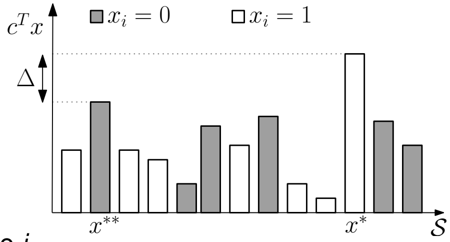

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- Union Bound over all n choices for i. □

Extensions

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[Beier, Vöcking (STOC 2004)]

Theorem remains true if linear constraints are perturbed.

[R., Vöcking (IPCO 2005)]

Theorem remains true for integer optimization problems.

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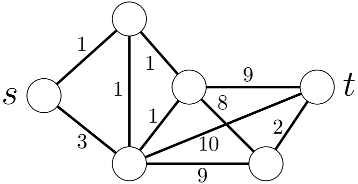
2. **Multiobjective Optimization**
   How many Pareto-optimal solutions do usually exist?
3. **Conclusions**

Single-criterion Optimization Problem: min $f(x)$ subject to $x \in \mathcal{S}$.

Example:
Shortest Path Problem



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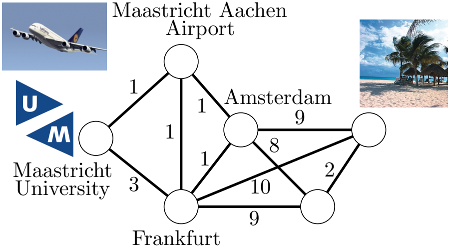

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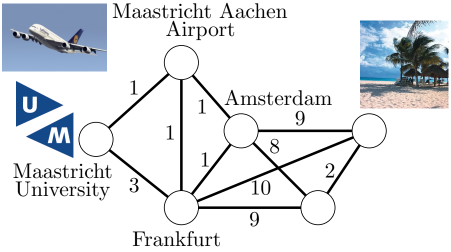

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Multiobjective Opt. Problem: min $f_1(x), \ldots,$ min $f_d(x)$ s.t. $x \in \mathcal{S}$.
Usually, there is no solution that is simultaneously optimal for all f_i.

Question

What can we do algorithmically to support the decision maker?

Pareto-optimal Solutions

Multiobjective Opt. Problem: $\min w^1(x), \ldots, \min w^d(x)$ s.t. $x \in \mathcal{S}$

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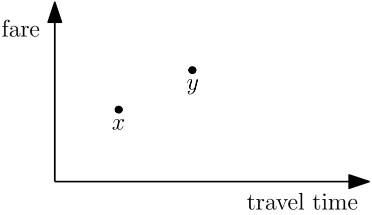

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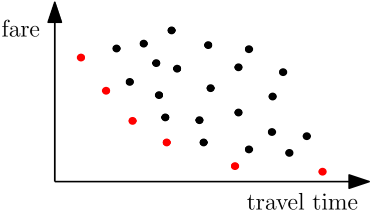

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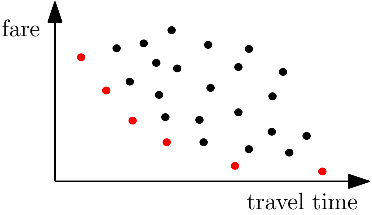

Often the **Pareto curve** is generated:

- Pareto curve limits options for decision maker.
- Monotone functions are optimized by Pareto-optimal solutions, e.g., $\lambda_1 w^1(x) + \ldots + \lambda_d w^d(x)$ or $w^1(x) \cdot \cdots \cdot w^d(x)$.
- Tool for solving single-criterion problems

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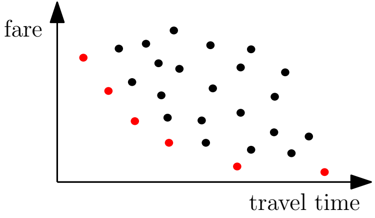

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Central Question

How large is the Pareto curve?

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- d linear objective functions:
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  solution $x = (x_1, \ldots, x_n) \in \mathcal{S}$ consists of n binary variables
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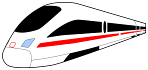
  $\forall i \in \{1, \ldots, d\}$: $\min w^i(x) = w^i_1 x_1 + \cdots + w^i_n x_n$

How large is the Pareto curve?

- Exponential in the worst case for almost all problems.
- In practice, often few Pareto optimal solutions.



Example: Train Connections
w.r.t. travel time, fare, number of train changes
[Müller-Hannemann, Weihe 2001]

Results (Bicriteria Optimization)

Adversary chooses $\mathcal{S}$ and a probability density $f_j^i \colon [-1,1] \to [0,\phi]$ for every w_j^i and some $\phi \geq 1$.
Every w_j^i is **drawn independently** according to f_j^i.

$$P_d(n,\phi) = \max_{\mathcal{S}, f_j^i} \mathbf{E}\left[\text{number of Pareto-optimal sol. for } \mathcal{S} \text{ and } f_j^i\right]$$

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extension to integer optimization problems

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Theorem [Brunsch, R. (TAMC 2011, STOC 2012)]

$P_d(n, \phi) = O(n^{2d}\phi^{d})$ $\quad$ $P_d(n, \phi) = \Omega(n^{d-1.5}\phi^{d})$

extension to non-linear objective functions

Bicriteria Optimization

Beier, R., Vöcking (IPCO 2007)

- min $w^1(x) = w_1x_1 + \cdots + w_nx_n$ and min $w^2(x)$
- subject to $x \in \mathcal{S} \subseteq \{0,1\}^n$, $\mathcal{S}$ arbitrary
- w_j drawn according to $f_j \colon [0,1] \to [0,\phi]$ for $\phi \geq 1$

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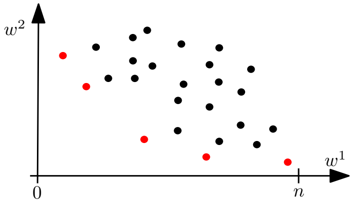

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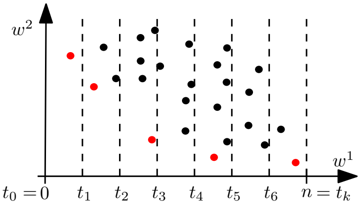

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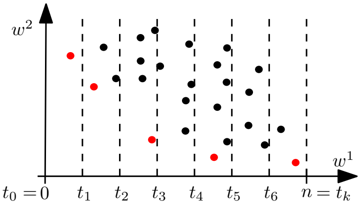

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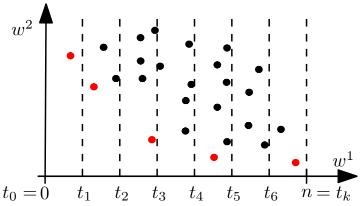

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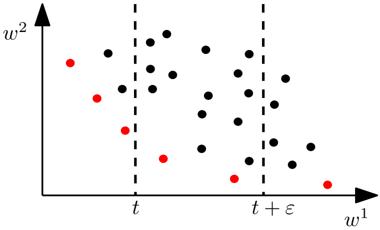

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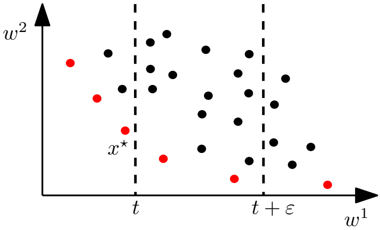

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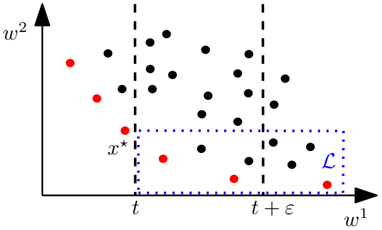

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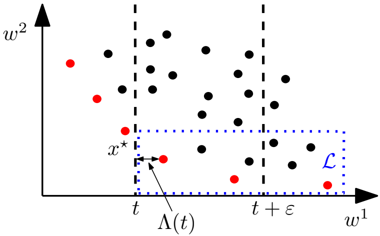

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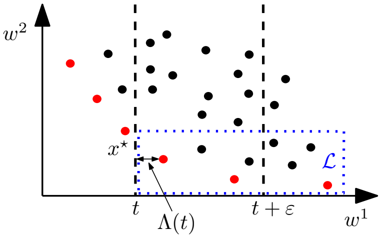

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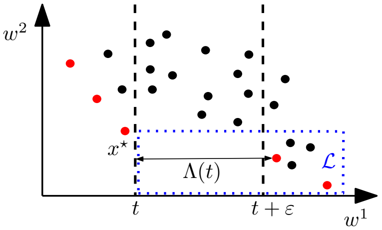

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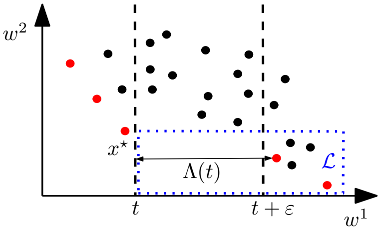

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Lemma [Beier, Vöcking (STOC 2004)]

For every $\varepsilon \geq 0$ and $t \in \mathbb{R}$, **Pr** $[\Lambda(t) \leq \varepsilon] \leq n\phi\varepsilon$.

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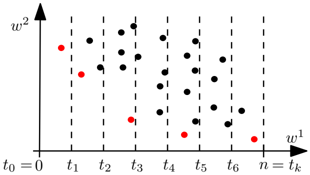

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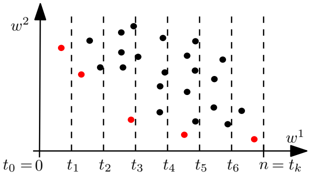

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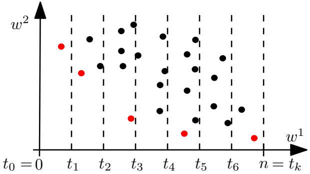
\end{aligned}
$$



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Outline

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   When does a binary optimization problem have polynomial smoothed complexity?
2. **Multiobjective Optimization**
   How many Pareto-optimal solutions do usually exist?
3. **Conclusions**

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Open Questions

- analyze other pivot rules for simplex method
- improve exponents of smoothed running time for 2-Opt etc.
- analyze your favorite problem/algo that is hard in the worst case
- use insights to develop better algorithms
- explore other frameworks for realistic theory