# Dynamic Allocation and Pricing: A Mechanism Design Approach

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# Modern Revenue Management

- Starts with US Airline Deregulation Act of 1978.
- Today mainstream business practice (airlines, trains, hotels, car rentals, holiday resorts, advertising, intelligent metering devices, etc..)
- Considerable gap between practitioners and academics in the field
- Major academic textbook : *The Theory and Practice of Revenue Management* by K. T. Talluri and G.J. van Ryzin

# Basic RM Questions (Talluri & van Ryzin)

- Quantity decisions: How to allocate capacity/output to different segments, products or channels ? When to withhold products from the market ?
- Structural decisions: Which selling format ? (posted prices, negotiations, auctions, etc..). Which features for particular format ?(segmentation, volume discounts, bundling, etc..)
- Pricing decisions: How to set posted prices, reserve prices ? How to price differentiate ? How to price over time ? How to markdown over life time ?

# Towards a Modern Theory of RM

- Necessary blend of

1. The elegant **dynamic models** from the OR, MS, CS , Econ (search) literatures with historical focus on "**grand, centralized optimization**" and/or "**ad-hoc**", intuitive mechanisms.
2. The rich, classical mechanism design literature with historical focus on **information/incentives** in **static** settings.

- Blend fruitful for numerous applications.

# Reviewed Papers (joint work with A. Gershkov)

- Dynamic Revenue Maximization with Heterogeneous Objects: A Mechanism Design Approach, *AEJ: Microeconomics 2009*
- Efficient Sequential Assignment with Incomplete Information,*GEB 2010*
- Revenue Maximization for the Dynamic Knapsack Problem, *TE* 2011
- Learning About The Future and Dynamic Efficiency, *AER 2009*
- Optimal Search, Learning, and Implementation, *mimeo 2010.*
- Efficient Dynamic Allocation with Strategic Arrivals, *mimeo 2011*

# The Gallego & van Ryzin Model (*MS 1994*) I

- Revenue Maximizing (RM) seller has $n$ identical objects that perish after deadline $T$.
- Agents arrive according to a Poisson process with intensity $\lambda$.
- Agents' values are private information, represented by I.I.D random variables $X_i = X$ on $[0, +\infty)$ with common c.d.f. $F$.
- Agents desire one object only, and can only be served upon arrival (no recall)
- After an item is assigned, it cannot be reallocated.

# Gallego & van Ryzin II

- Gallego & van Ryzin restrict attention to posted prices: at each point in time $t$, seller sets price $p_t$ that needs to be paid by any buyer that arrives at $t$.
- **Main Results:**
  1. Optimal posted-price revenue is concave in the number of objects, and in time until deadline.
  2. Time pattern of optimal prices: Downward trend, interrupted by upward jumps after each sale.
  3. Fixed price is approximately optimal if $\min(n, \lambda T)$ is large enough.

# New Research Questions

- General Mechanisms
- Multiple, Heterogenous Objects
- Multi-Unit Demand
- Learning about Demand
- Recall and Strategic Arrivals

# Albright's Model (*MS 1974*) I

- Welfare Maximizing (WM) seller has $m$ items.
- Each item $i = 1, .., n$ is characterized by a "quality" $q_i$ with

$$0 \leq q_n \leq q_{n-1} \leq ... \leq q_1$$

- If an item with quality $q_i \geq 0$ is assigned to an agent with type $x_j$ this agent has utility $q_i x_j$
- Complete information: current type is known upon arrival; future types are IID random variables $X_i = X$ on $[0, +\infty)$ with common c.d.f. $F$.
- Poisson arrivals, unit demand, deadline, etc..as in Gallego & van Ryzin.

# The Welfare Maximizing (WM) Allocation

## Theorem (Albright)

*Denote by $\Pi_t$ the set of items available at $t$, with cardinality $k_t$. There exist $n+1$ unique functions*

$$\infty \equiv y_0(t) \geq y_1(t) \geq ..y_n(t) \geq 0, \ \forall t$$

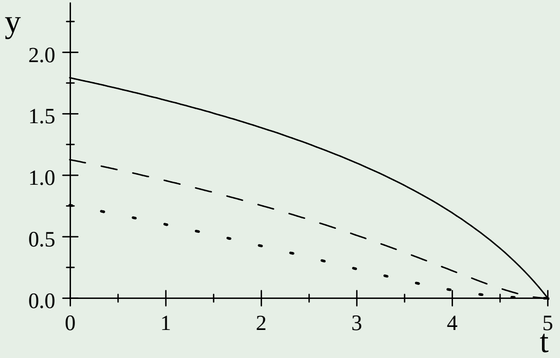
*which* **do not depend on the $q$'s** *such that if an agent with type $x$ arrives at time $t$, it is optimal to assign him the $j$'th highest element of $\Pi_t$ if $x \in [y_j(t), \ y_{j-1}(t))$ and not to assign any object if $x < y_{k_t}(t)$. For each $k$, the function $y_k(t)$ satisfies*

$$y_k'(t) = -\lambda \int_{y_k}^{y_{k-1}} (1 - F(x))dx.$$

# Albright: Illustration.

## Example

There are three objects; $\lambda = 1$; the distribution of agents' types is $F(x) = 1 - e^{-x}$. The following figure depicts the solution for $T = 5$ :



# Mechanisms & Implementation with Incomplete Info. I

- Assume types are private information. If an item with quality $q_i \geq 0$ is assigned to an agent with type $x_j$ for price $p_i$, this agent has utility $q_i x_j - p_i$.
- W.l.o.g. restrict attention to deterministic, Markovian and direct mechanisms where every agent, upon arrival, reports his characteristic $x_i$ and where, at any point in time $t$, the mechanism specifies:

1. a non-random allocation rule (which object is allocated, if any) that only depends on $t$, on the declared type of the arriving agent, and on the inventory of items available at $t$.
2. a payment to be made by the arriving agent which depends on $t$, on the declared type of the agent, and on the inventory of items available at $t$.

# Mechanisms & Implementation with Incomplete Info. II

## Theorem

**1)** *An allocation policy* $\{Q_t(x, \Pi_t)\}_t$ *is implementable iff at each* $t$ *it partitions the set of agents' types into* $k_t + 1$ *disjoint intervals such that all types in a given interval obtain the same quality, and such that higher types obtain a higher quality.*

**2)** *The associated payment scheme is given by*

$$P_t(x, \Pi_t) = \sum_{i=j}^{k_t} (q_{(i:\Pi_t)} - q_{(i+1:\Pi_t)}) y_{i,\Pi_t}(t) \text{ if } x \in [y_{j,\Pi_t}(t), y_{j-1,\Pi_t}(t))$$

*and by zero otherwise.*

- The WM policy is implementable. Payments: dynamic analogue of the Vickrey-Clarke-Groves mechanism (see also Bergemann and Valimäki, *EC 10)*

# RM with Heterogenous Objects

## Theorem

*Assume that the virtual value $x - \frac{1-F(x)}{f(x)}$ is increasing. Then the RM allocation is given by $n$ cut-off functions that do not depend on the available qualities. These functions satisfy:*

$$y_i(t) - \frac{1-F(y_i(t))}{f(y_i(t))} + \lambda \int_t^T \frac{[1-F(y_{i-1}(s))]^2}{f(y_{i-1}(s))} ds = \lambda \int_t^T \frac{[1-F(y_i(s))]^2}{f(y_i(s))} d$$

*or, equivalently*

$$y_i(t) - \frac{1-F(y_i(t))}{f(y_i(t))} + R(\mathbf{1}_{i-1}, t) = R(\mathbf{1}_i, t)$$

*where $R(\mathbf{1}_j, t)$ is the expected revenue at time $t$ from the optimal policy if $j$ identical objects with $q = 1$ are still available at that time.*

# Clearance Sales

- Percentage Markdown: Difference between prices of the same product at $t = 0$ and $t = T$, divided by the price at $t = 0$.
- Pashigian and Bowen (*QJE 91*) empirically find that :

*"More expensive apparel items within each product line are frequently sold at a higher average percentage markdown"*

**Theorem**

*Assume an RM seller and consider the scenario where at time $t = 0$ there are $n_1 > 0$ items of quality $q$ and $n_2 > 0$ items of quality $s < q$, while at time $t = T$ there are $l_1 > 0$ items of quality $q$ , and $l_2 > 0$ items of quality $s$ left unsold. Then the percentage markdown is always higher for the higher quality.*

# Optimal Inventory Design

## Theorem

*Let* $y = \{y_i(t)\}_{i=1}^n$ *denote the allocation underlying the RM policy with* $n$ *objects, and assume that the cost of producing qualities* $(q_1, q_2, ..q_n)$ *is given by* $C(q_1, q_2, ..q_n) = \sum_{i=1}^n \phi(q_i)$ *where* $\phi : R \longrightarrow R$ *is strictly increasing, convex and satisfies* $\phi(0) = 0$. *Then*
*1) The optimal number of objects* $n^*$ *is characterized by*

$$\phi^{'}(0) \in (y_{n^*+1}(0) - \frac{1 - F(y_{n^*+1}(0))}{f(y_{n^*+1}(0))},\; y_{n^*}(0) - \frac{1 - F(y_{n^*}(0))}{f(y_{n^*}(0))}]$$

*2) The optimal qualities* $q_i^*$ *are given by:*

$$\phi^{'}(q_i^*) = y_i(0) - \frac{1 - F(y_i(0))}{f(y_i(0))},\; i = 1, ..., n^*$$

# Multi-Unit Demand: Dynamic Knapsack

- An RM seller has capacity $C \in \mathbb{R}_+$ that perishes after $T$ periods.
- In each period, impatient agent arrives with quantity request $w$, and per-unit value $v$. Type $(w, v)$ is private information to the arriving agent.
- Type $(w, v)$'s utility is given by $wv - p$ if at price $p$ he is allocated a capacity $w' \geq w$ and by $-p$ if he is assigned an insufficient capacity $w' < w$.
- Demands are I.I.D across periods, governed by c.d.f. $F(w, v)$ with density $f(w, v) > 0$ on $[0, \infty)^2$. For all $w$, the conditional virtual value $v - \frac{1-F(v|w)}{f(v|w)}$ is an unbounded, strictly monotone function of $v$.
- Complete information optimization model due to Kleywegt & Papastavrou, *OR 2001;* they do not consider payments.

# Dynamic Knapsack: Implementable Policies

- A deterministic, Markovian allocation rule for time $t$ with remaining capacity $c$ has the form $\alpha_t^c : [0, +\infty)^2 \to \{1, 0\}$ where 1 (0) means that the reported capacity demand $w$ is satisfied (not satisfied).

## Theorem

*A policy* $\{\alpha_t^c\}_{t,c}$ *is implementable iff for every* $t$ *and every* $c$ *it satisfies:*
*1)* $\forall\,(w, v), v' \geq v,\ \alpha_t^c(w, v) = 1 \Rightarrow \alpha_t^c(w, v') = 1.$
*2) The function* $wp_t^c(w)$ *is non-decreasing in* $w$*, where*
$p_t^c(w) = \inf\{v\ /\ \alpha_t^c(w, v) = 1\}.$
*The maximal, individually rational payment function that implements* $\{\alpha_t^c\}_{t,c}$ *is given by*

$$q_t^c(w, v) = \begin{cases} wp_t^c(w) & \text{if } \alpha_t^c(w, v) = 1 \\ 0 & \text{if } \alpha_t^c(w, v) = 0 \end{cases}$$

# Dynamic Knapsack: RM I

## Theorem

*Assume that:*

1. *For any $w$, the hazard rate $\frac{f(v|w)}{1-F(v|w)}$ is non-decreasing in $v$.*
2. *For any $w' \geq w$, and for any $v$, $\frac{f(v|w)}{1-F(v|w)} \geq \frac{f(v|w')}{1-F(v|w')}$.*
   *For each $c, t, w$ let $p_t^c(w)$ denote the recursive solution to the system*

$$w\left(p_t^c(w) - \frac{1-F(p_t^c(w)|w)}{f(p_t^c(w)|w)}\right) = R^*(c, T-t) - R^*(c-w, T-t).$$

*where $R^*$ denotes the optimal revenue with $R^*(c,0) = 0$ for all $c$. Then the underlying allocation where $\alpha_t^c(w,v) = 1$ iff if $v \geq p_t^c(w)$ is implementable. In particular, the above system determines the RM policy.*

# Dynamic Knapsack: RM II

- Optimal revenue may not be concave in capacity (see Kleywegt & Papastavrou, *OR 2001* for non-concavity in WM) - this is connected to implementation problems.
- Under a concavity conditions on the distribution of types, revenue is concave and above system also determines RM policy.
- RM policy requires price adjustments for every $c, t, w$ - complicated dynamics. We construct a static nonlinear price schedule (it uses correlations between $w$ and $v$ !) that is asymptotically optimal if $\min(C, T)$ is large enough.

# Learning About Demand: Example

- One object with quality $q = 1$;
- Two privately informed agents arrive sequentially, one per period.
- Each agent is impatient and can only be served upon arrival.
- WM designer does not know the distribution of types (values): with probability 0.5 (0.5) she believes that agents' values are independently drawn from $U[0,1]$ $(U[1,2])$.

# Efficient Allocation (First-Best) Is Not "Online" Implementable

- After observing $x_1 < (>) 1$, $x_2$ is known to be $U[0,1]$ $(U([1,2])$.
- This yields $E(x_2/x_1) = \begin{cases} 0.5 & \text{if} \quad x_1 < 1 \\ 1 & \text{if} \quad x_1 = 1 \\ 1.5 & \text{if} \quad x_1 > 1 \end{cases}$
- Thus, under complete information, the first arriving agent should get the object if $x_1 \in [0.5, 1] \cup [1.5, 2]$.
- The above allocation is not monotone, and therefore not implementable under incomplete information about types.
- "Offline" implementation is possible.

# When is First-Best Implementable ?

**Theorem**

*Assume that, for any history of observations, the complete information WM cutoffs are 1-Lipschitz functions of the current observations. Then, the dynamic WM policy is implementable under incomplete information.*

- Insight taken from the theory of mechanism design with interdependent values (Jehiel & Moldovanu, *EC 01*)
- Above condition can be translated into explicit, sufficient conditions on the model's primitives (designer's prior, learning process)
- Results generalize insights about the *reservation price property* obtained in search models, due to Rothschild (*JPE 74*), Albright (*MS 77*), Rosenfield & Shapiro (*JET 81*).

# Characterization of Incentive-Efficient (Second-Best) Mechanism

**Theorem**

*1) The second-best mechanism is deterministic. That is, for every history, and for every type of arriving agent, there exists a quality $q$ that is allocated to that agent with probability* $1$ *.*
*2) At each period, the optimal mechanism partitions the type set of the arriving agent into a collection of disjoint intervals such that all types in a given interval obtain the same quality with probability* $1$*, and such that higher types obtain a higher quality.*

- Proof uses insights due to Riley & Zeckhauser (*QJE 83*) and concepts from *majorization/Schur convexity* (see Marschall & Olkin, *79*).

# Strategic Arrivals and Recall

- Designer endowed with indivisible object;
- Stream of randomly arriving, **long-lived** agents; arrivals described by a counting process $\{\mathcal{N}(t), t \geq 0\}$ in continuous time
- Agent's private information is two-dimensional: arrival time $t \geq 0$ and value $v \geq 0$ for the object. Arrival time and value are independent of each other.
- Values are I.I.D. random variables with common c.d.f. $F$, continuos density $f$.
- If an agent arrives at $t$, gets the object at $\tau \geq t$ and pays $p$ at $\tau' \in [t, \tau]$, then her utility is given by $e^{-\delta\tau}v - e^{-\delta\tau'}p$ where $\delta \in (0,1)$ is the discount factor.
- Designer's utility is given by $e^{-\delta\tau}v$ (WM) or $e^{-\delta\tau'}p$ (RM).

# Renewals (Zuckerman, *JEDC 88*, Gallien, *OR 06*)

- A non-negative random variable $W$ satisfies *NBU* (new better than old) if, for every $y > 0$, $W$ is stochastically larger than the conditional random variable $(W - y / W \geq y)$.

**Theorem**

*Assume that the arrival process is a renewal with inter-arrival distribution $G$ (and Laplace Transform $\phi$) that satisfies NBU. Assume also that $x - \frac{1-F(x)}{f(x)}$ is strictly increasing, and let $H$ denote the distribution of virtual values. Then, revenue is maximized by charging a constant price $P$, where $P$ is the unique solution to*

$$p = \frac{\phi(\delta)}{1 - \phi(\delta)} \int_p^\infty (z - p) dH(z) dz$$

*In particular, recall is never used, and all allocations occur upon arrival.*

# Learning from Arrivals: Example

- Distribution of values is $U\left[0,1\right]$.
- Inter-arrival time is known: 1) $U[1,2]$ or 2) $U\left[2,3\right]$.
- The optimal cutoffs for a WM designer are

$$x_{[i,j]}(\delta) = \frac{1}{\phi_{[i,j]}\left(\delta\right)}\left(1 - \sqrt{1 - \left(\phi_{[i,j]}\left(\delta\right)\right)^2}\right)$$

where

$$\phi_{[i,j]}\left(\delta\right) = \frac{e^{-i\delta} - e^{-j\delta}}{\delta}.$$

is the respective Laplace transform.

- Fix any $\delta$ and note that $x_{[1,2]} = x_{[1,2]}\left(\delta\right) > x_{[2,3]}\left(\delta\right) = x_{[2,3]}$.

# Unknown Arrival Process / Complete Information

- Designer does not know the distribution of inter-arrival times: believes that it is either $U[1,2]$ or $U[2,3]$, with equal probabilities.
- Let $t_1$ be the time of the first arrival. Under complete information about types the optimal policy is:

1. At time $t \in (1,2]$ the cutoff is $x_{[1,2]}$
2. At time $t > 2$ the cutoff is $x_{[1,2]}$ if $t_1 < 2$, and it is $x_{[2,3]}$ otherwise.

# Unknown Arrival Process / Incomplete Information

- The allocative externality payment, which needs to be paid for the object by an agent arriving at $t \geq t_1$ is

$$P(t_1) = \begin{cases} x_{[1,2]} & \text{if } t_1 \in [1,2] \\ x_{[2,3]} & \text{if } t_1 \in (2,3] \end{cases} .$$

- Consider type $(t, v)$ with $t \in (1,2)$ and $v \in (x_{[2,3]}, x_{[1,2]})$. Truthful reporting yields zero utility since object not allocated to him. But, a report $(t', v)$ where $t' = t + 1 \in (2,3)$ yields utility $e^{-\delta t'}\left(v - x_{[2,3]}\right) > 0$ .
- Truthful reporting under standard Clarke-Groves-Vickrey mechanism is not optimal, and WM dynamic allocation cannot be implemented !
- Problem: Informational externality on designer induced by early arrivals.

# Subsidizing Early Arrivals

- Subsidy paid to agent that arrives at $t$, independently of whether he gets the object

$$S(t) = \begin{cases} x_{[1,2]} - x_{[2,3]} > 0 & \text{if } t \in [1,2] \\ 0 & \text{if } t \geq 2 \end{cases} .$$

- Combination of externality payment made by the winner together with the subsidy scheme does implement the WM dynamic allocation **in general**.
- Consider *winner pay only* mechanisms: agents that do not get the object pay nothing. No payment scheme in this class can implement the WM allocation in the example, and in more general settings where later arrivals make the designer more pessimistic.
- Special settings exists where subsidy is not needed, e.g., renewals.

# Long Lived Agents: Additional Topics

- Long-lived agents, finite horizon, no learning (Board & Skrypacz, 10)
- Strategic deadlines (Pai & Vohra 08, Mierendorff 09)
- Queueing (Dolan, *Bell 78*, Hassin & Haviv 02, Kittsteiner & Moldovanu, *MS* 05)
- Changing types over time (see Pavan, Segal & Toikka 09)
- Short lived objects (Said 09)

# Conclusion

- Modern theory of revenue management.
- Fruitful blend of

1. Dynamic models from the OR, MS, CS , Econ (search) literatures with focus on "grand, centralized optimization" and/or "ad-hoc", intuitive mechanisms.
2. Mechanism design literature with focus on information/incentives in static settings.

- Much remains to be done, e.g., competition and dynamic pricing.