Optimization with Copositive and Completely Positive Matrices

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Outline

1. Copositive and Completely Positive Matrices
   - Background
   - Applications in Optimization

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   - Separation for matrices with block structure
   - Applications

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   - Approximation hierarchies for CoP
   - Approximation hierarchies for CP

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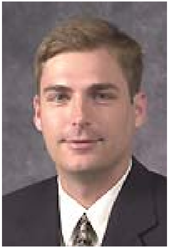
   - Approximation hierarchies for CoP
   - Approximation hierarchies for CP
4. Open Problems

Immanuel Bomze (Vienna)



Sam Burer (Iowa)

Etienne de Klerk (Tilburg)

Monique Laurent (CWI/Tilburg)

Mirjam Duer (Trier)

Franz Rendl (Klagenfurt)

Outline

Let $\mathcal{S}_n$ denote the set of $n \times n$ real symmetric matrices, $\mathcal{S}_n^+$ denote the cone of $n \times n$ real symmetric positive semidefinite matrices and $\mathcal{N}_n$ denote the cone of symmetric nonnegative $n \times n$ matrices.

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- The cone of $n \times n$ completely positive (CP) matrices is $\mathcal{C}_n = \{X \,|\, X = AA^T$ for some $n \times k$ nonnegative matrix $A\}$.
- Dual of $\mathcal{C}_n$ is the cone of $n \times n$ copositive (CoP) matrices, $\mathcal{C}_n^* = \{X \in \mathcal{S}_n \,|\, y^T Xy \geq 0 \; \forall \, y \in \mathbb{R}_+^n\}$.

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Extensive linear algebra literature for CP and CoP matrices; see for example new survey article on CoP matrices by Hiriart-Urruty and Seeger in *SIAM Review* (December, 2010).

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- Literature on CP matrices largely concerned with issue of CP-rank (minimum k so that $X = AA^T$ for some $n \times k$ nonnegative A).
- One relevant topic concerns the distinction between completely positive and doubly nonnegative (DNN) matrices. The cone of $n \times n$ DNN matrices is $\mathcal{D}_n = \mathcal{S}_n^+ \cap \mathcal{N}_n$. Clear that $\mathcal{C}_n \subset \mathcal{D}_n$.

CP Graphs

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Let G be an undirected graph on n vertices. Then G is called a CP graph if any matrix $X \in \mathcal{D}_n$ with $G(X) = G$ also has $X \in \mathcal{C}_n$.

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Immediately implies that for $n \leq 4$,

$$\mathcal{C}_n = \mathcal{D}_n, \qquad \mathcal{C}_n^* = \mathcal{D}_n^* = \mathcal{S}_n^+ + \mathcal{N}_n.$$

Outline

- Lemke's algorithm for the LCP $Iw - Mz = q$, $w \geq 0$, $z \geq 0$, $w^T z = 0$ converges if M is a "copositive plus" (but not generally symmetric) matrix.

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- Nonconvex quadratic optimization over the simplex ("standard QP") (Bomze et al., 2000)
- Computing a maximum stable set or maximum clique in a graph (DeKlerk and Pasechnik, 2002)
- The quadratic assignment problem (Povh and Rendl, 2009)

Result of Burer (2009) shows broad applicability for CP/CoP matrices in optimization. Consider problem

$$\begin{array}{lll} \text{(MIQP)} & \min & x^T Qx + c^T x \\ & \text{s.t.} & Ax = b \\ & & x \geq 0, \quad x_i \in \{0,1\}, i \in \mathcal{B}, \end{array}$$

where A is an $m \times n$ matrix and $\mathcal{B} \subset \{1, 2, \ldots, n\}$.

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\end{aligned}
$$

where A is an $m \times n$ matrix and $\mathcal{B} \subset \{1, 2, \ldots, n\}$. Let

$$Y = Y(x, X) = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

Theorem (CP representation of MIQP)

Assume that MIQP is feasible and the solution set is bounded. Then the solution value in MIQP is equal to the solution value for the problem

$$\begin{aligned} \min \quad & Q \bullet X + c^T x \\ \text{s.t.} \quad & Ax = b \\ & a_i^T X a_i = b_i^2, \quad i = 1, \ldots, m \\ & Y \in \mathcal{C}_{n+1}, \;\; X_{ii} = x_i, \; i \in \mathcal{B}. \end{aligned}$$

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Given a matrix $X \in \mathcal{D}_n \setminus \mathcal{C}_n$, want to separate X from $\mathcal{C}_n$ using a matrix $V \in \mathcal{C}_n^*$ having $V \bullet X < 0$.

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- $\mathcal{D}_n$ is a tractable relaxation of $\mathcal{C}_n$. Expect that solution of relaxed problem will be $X \in \mathcal{D}_n \setminus \mathcal{C}_n$.
- Note that least n where problem occurs is $n = 5$.

- Known that extreme rays of $\mathcal{D}_5$ are either rank-one matrices in $\mathcal{C}_5$, or rank-three "extremely bad" matrices where $G(X)$ is a 5-cycle (every vertex in $G(X)$ has degree two).

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- Burer, A. and Dür (2009) show that any such extremely bad matrix can be separated from $\mathcal{C}_5$ by a transformation of the Horn matrix

$$H := \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix} \in \mathcal{C}_5^* \setminus \mathcal{D}_5^*.$$

Dong and A. (2010) show that:

- Separation procedure based on transformed Horn matrix applies to $X \in \mathcal{D}_5 \setminus \mathcal{C}_5$ where X has rank three and $G(X)$ has *at least one* vertex of degree 2.

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We will describe the procedure from Dong and A. (2010) for $X \in \mathcal{D}_5 \setminus \mathcal{C}_5,\ X \not> 0$ and its generalization to larger matrices having block structure.

Assume that $X \in \mathcal{D}_5$, $X \not\succ 0$. After a symmetric permutation and diagonal scaling, X may be assumed to have the form

$$X = \begin{pmatrix} X_{11} & \alpha_1 & \alpha_2 \\ \alpha_1^T & 1 & 0 \\ \alpha_2^T & 0 & 1 \end{pmatrix}, \tag{1}$$

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where $X_{11} \in \mathcal{D}_3$.

Theorem (Berman and Xu, 2004)

Let $X \in \mathcal{D}_5$ have the form (1). Then $X \in \mathcal{C}_5$ if and only if there are matrices A_{11} and A_{22} such that $X_{11} = A_{11} + A_{22}$, and

$$\begin{pmatrix} A_{ii} & \alpha_i \\ \alpha_i^T & 1 \end{pmatrix} \in \mathcal{D}_4, \quad i = 1, 2.$$

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Theorem (Generation of cut, 5×5 case)

Assume that $X \in \mathcal{D}_5$ has the form (1). Then $X \in \mathcal{D}_5 \setminus \mathcal{C}_5$ if and only if there is a matrix

$$V = \begin{pmatrix} V_{11} & \beta_1 & \beta_2 \\ \beta_1^T & \gamma_1 & 0 \\ \beta_2^T & 0 & \gamma_2 \end{pmatrix} \quad \textit{such that} \quad \begin{pmatrix} V_{11} & \beta_i \\ \beta_i^T & \gamma_i \end{pmatrix} \in \mathcal{D}_4^*, \quad i = 1, 2,$$

and $V \bullet X < 0$.

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- Suppose that $X \in \mathcal{D}_5 \setminus \mathcal{C}_5$, and V is a matrix that satisfies the conditions of the previous Theorem. If $\tilde{X} \in \mathcal{C}_5$ is another matrix of the form (1), then know that $V \bullet \tilde{X} \geq 0$.

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- However, cannot conclude that $V \in \mathcal{C}_5^*$ because $V \bullet \tilde{X} \geq 0$ only holds for $\tilde{X}$ of the form (1), in particular, $\tilde{x}_{45} = 0$.

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- However, cannot conclude that $V \in \mathcal{C}_5^*$ because $V \bullet \tilde{X} \geq 0$ only holds for $\tilde{X}$ of the form (1), in particular, $\tilde{x}_{45} = 0$.
- Fortunately, result of Hogben, Johnson and Reams (2005) shows that V can easily be "completed" to obtain a copositive matrix that still separates X from $\mathcal{C}_5$.

Theorem (Completion of cut, 5×5 case)

Suppose that $X \in \mathcal{D}_5 \setminus \mathcal{C}_5$ has the form (1), and that V satisfies the conditions of the previous theorem. Define

$$V(s) = \begin{pmatrix} V_{11} & \beta_1 & \beta_2 \\ \beta_1^T & \gamma_1 & s \\ \beta_2^T & s & \gamma_2 \end{pmatrix}.$$

Then $V(s) \bullet X < 0$ for any s, and $V(s) \in \mathcal{C}_5^*$ for $s \geq \sqrt{\gamma_1 \gamma_2}$.

Outline

Procedure for 5×5 case where $X \not\succeq 0$ can be generalized to larger matrices with block structure. Assume X has the form

$$X = \begin{pmatrix} X_{11} & X_{12} & X_{13} & \dots & X_{1k} \\ X_{12}^T & X_{22} & 0 & \dots & 0 \\ X_{13}^T & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ X_{1k}^T & 0 & \dots & 0 & X_{kk} \end{pmatrix}, \tag{2}$$

where $k \geq 3$, each X_{ii} is an $n_i \times n_i$ matrix, and $\sum_{i=1}^{k} n_i = n$.

Lemma (Characterization of CP matrix with block structure)

Suppose that $X \in \mathcal{D}_n$ *has the form* (2), $k \geq 3$, *and let*

$$X^i = \begin{pmatrix} X_{11} & X_{1i} \\ X_{1i}^T & X_{ii} \end{pmatrix}, i = 2, \ldots, k.$$

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Then $X \in \mathcal{C}_n$ *if and only if there are matrices* A_{ii}, $i = 2, \ldots, k$ *such that* $\sum_{i=2}^k A_{ii} = X_{11}$, *and*

$$\begin{pmatrix} A_{ii} & X_{1i} \\ X_{1i}^T & X_{ii} \end{pmatrix} \in \mathcal{C}_{n_1+n_i}, \quad i = 2, \ldots, k.$$

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$$\begin{pmatrix} A_{ii} & X_{1i} \\ X_{1i}^T & X_{ii} \end{pmatrix} \in \mathcal{C}_{n_1+n_i}, \quad i = 2, \ldots, k.$$

Moreover, if $G(X^i)$ is a CP graph for each $i = 2, \ldots, k$, then the above statement remains true with $\mathcal{C}_{n_1+n_i}$ replaced by $\mathcal{D}_{n_1+n_i}$.

Theorem (Existence of cut, block case)

Suppose that $X \in \mathcal{D}_n \setminus \mathcal{C}_n$ has the form (2), where $G(X^i)$ is a CP graph, $i = 2, \ldots, k$. Then there is a matrix V, also of the form (2), such that

$$\begin{pmatrix} V_{11} & V_{1i} \\ V_{1i}^T & V_{ii} \end{pmatrix} \in \mathcal{D}^*_{n_1+n_i}, \quad i = 2, \ldots, k,$$

and $V \bullet X < 0$.

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and $V \bullet X < 0$. Moreover, if $\gamma_i = [\mathrm{diag}(V_{ii})]^{.5}$, then the matrix

$$\tilde{V} = \begin{pmatrix} V_{11} & \ldots & V_{1k} \\ \vdots & \ddots & \vdots \\ V_{1k}^T & \ldots & V_{kk} \end{pmatrix},$$

where $V_{ij} = \gamma_i \gamma_j^T$, $2 \le i \ne j \le k$, has $\tilde{V} \in \mathcal{C}^*_n$ and $\tilde{V} \bullet X = V \bullet X < 0$.

Note that:

- Matrix X may have numerical entries that are small but not exactly zero. Can apply procedure to perturbed matrix $\tilde{X}$ where entries of X below a specified tolerance are set to zero. If a cut V separating $\tilde{X}$ from $\mathcal{C}_n$ is found, then $V \bullet X \approx V \bullet \tilde{X} < 0$, and V is very likely to also separate X from $\mathcal{C}_n$.

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- Procedure may provide a cut for $X \in \mathcal{D}_n \setminus \mathcal{C}_n$ even when sufficient conditions for generating such a cut are not satisfied. In particular, a cut may be found even when the condition that X^i is a CP graph for each i is not satisfied.

A second case where block structure can be used to generate cuts for a matrix $X \in \mathcal{D}_n \setminus \mathcal{C}_n$ is when X has the form

$$X = \begin{pmatrix} I & X_{12} & X_{13} & \dots & X_{1k} \\ X_{12}^T & I & X_{23} & \dots & X_{2k} \\ X_{13}^T & X_{23}^T & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & X_{(k-1)k} \\ X_{1k}^T & X_{2k}^T & \dots & X_{(k-1)k}^T & I \end{pmatrix}, \tag{3}$$

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where $k \geq 2$, each X_{ij} is an $n_i \times n_j$ matrix, and $\sum_{i=1}^k n_i = n$. The structure in (3) corresponds to a partitioning of the vertices $\{1, 2, \dots, n\}$ into k stable sets in $G(X)$, of size $n_1, \dots, n_k$.

Outline

Example (Stable set in a graph)

Let A be the adjacency matrix of a graph G on n vertices, and let α be the maximum size of a stable set in G.

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DeKlerk and Pasechnik (2002) show that

$$\alpha^{-1} = \min\left\{(I + A) \bullet X : ee^T \bullet X = 1, X \in \mathcal{C}_n\right\}. \tag{4}$$

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Relaxing $\mathcal{C}_n$ to $\mathcal{D}_n$ results in the Lovász-Schrijver bound

$$(\vartheta')^{-1} = \min\left\{(I + A) \bullet X : ee^T \bullet X = 1, X \in \mathcal{D}_n\right\}. \tag{5}$$

Let G_{12} be the complement of the graph corresponding to the vertices of a regular icosahedron (Bomze and DeKlerk, 2002). Then $\alpha = 3$ and $\vartheta' \approx 3.24$.

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- Using the cone $\mathcal{K}^1_{12}$ to better approximate the dual of (4) provides no improvement (Bomze and DeKlerk, 2002).
- For the solution matrix $X \in \mathcal{D}_{12}$ from (5), cannot find cut based on first block structure (2). However *can* find a cut based on second block structure (3), using four 3×3 diagonal blocks. Adding this cut and re-solving, gap to $1/\alpha = \frac{1}{3}$ is approximately 2×10^{-8}.

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- After adding all of the cuts found, re-solve the problem to get a new solution X^1 and a new bound on α. Continue for an additional three rounds of cuts, on each round i using the cuts obtained from $2n$ eligible structures, chosen at random, obtained from the solution of the previous problem X^{i-1}.

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Perform this entire procedure 20 times for each of the 3 problems.

Table: Results on stable set problems (20 runs for each problem)

Graph	α	ϑ'	ϑ^{cop}	Round	Number of cuts min	median	max	Bound values min	mean	max
G_{11}	4	4.694	4.280	1	13	16	19	4.342	4.362	4.443
				2	14	18	22	4.244	4.268	4.317
				3	12	17	21	4.237	4.253	4.279
				4	13	17	22	4.234	4.248	4.264
G_{14}	5	5.916	5.485	1	11	15	19	5.530	5.585	5.666
				2	12	17	22	5.441	5.479	5.548
				3	16	20	25	5.413	5.441	5.483
				4	14	17	25	5.405	5.422	5.456
G_{17}	6	7.134	6.657	1	7	12	18	6.731	6.814	6.922
				2	9	17	25	6.594	6.693	6.783
				3	10	15	23	6.571	6.651	6.718
				4	10	16	23	6.565	6.620	6.664

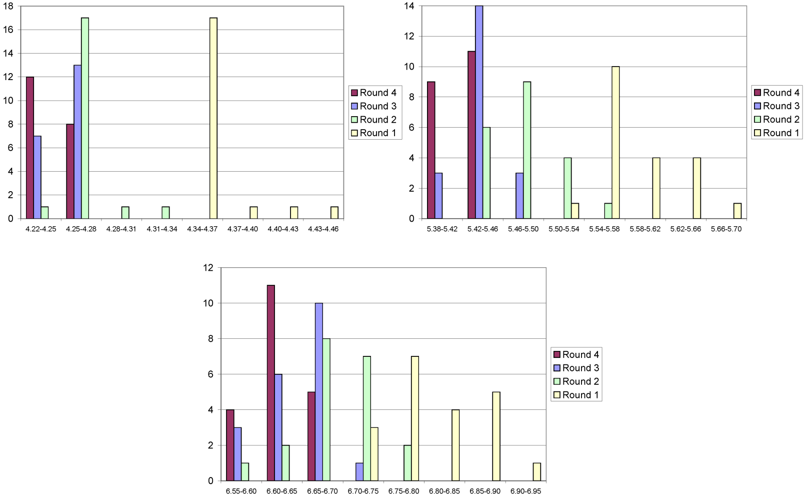

Figure: Bounds on max stable set for G_{11}, G_{14} and G_{17}

Outline

Clear that a matrix $M \in \mathcal{C}_n^*$ if and only if the polynomial

$$P^{(0)}(x) := (x \circ x)^T M (x \circ x) = \sum_{i=1}^{n} M_{ij} x_i^2 x_j^2$$

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is nonnegative for all $x \in \mathbb{R}^n$. This is obviously the case if $M \geq 0$.
A weaker sufficient condition that ensures nonegativity is if $P^{(0)}(x)$ can be written as a sum of squares (s.o.s.),

$$P^{(0)}(x) = \sum_{i=1}^{k} h_i(x)^2.$$

Parillo (2000) proved that $P^{(0)}(x)$ is a s.o.s. $\iff$ $M \in \mathcal{D}_n^* = \mathcal{S}_n^+ + \mathcal{N}_n$.

To generalize this construction, consider the polynomial

$$P^{(r)}(x) := P^{(0)}(x)\left(\sum_{i=1}^{n} x_i^2\right)^r = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \left(\sum_{i=1}^{n} x_i^2\right)^r.$$

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Using $P^{(r)}(x)$, define the inner approximation hierarchies

$$\begin{aligned}\mathcal{L}_n^r &:= \left\{M \mid P^{(r)}(x) \text{ has nonnegative coefficients}\right\},\\ \mathcal{K}_n^r &:= \left\{M \mid P^{(r)}(x) \text{ is a sum of squares}\right\}.\end{aligned}$$

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Easy to see that $\mathcal{L}_n^r \subseteq \mathcal{K}_n^r\ \forall r \in \mathbb{Z}_+$, and furthermore

$$\begin{aligned}\mathcal{N}_n = \mathcal{L}_n^0 \subseteq \mathcal{L}_n^1 \subseteq \cdots \subseteq \mathcal{L}_n^r \subseteq \mathcal{L}_n^{r+1} \subseteq \cdots \subseteq \mathcal{C}_n^*,\\ \mathcal{D}_n^* = \mathcal{K}_n^0 \subseteq \mathcal{K}_n^1 \subseteq \cdots \subseteq \mathcal{K}_n^r \subseteq \mathcal{K}_n^{r+1} \subseteq \cdots \subseteq \mathcal{C}_n^*.\end{aligned}$$

$\mathcal{L}_n^r$ and $\mathcal{K}_n^r$ are closed convex cones, and in fact $\mathcal{L}_n^r$ is polyhedral.

Let

$$\mathbb{I}_n(r) := \left\{ m \in \mathbb{Z}_+^n \mid e^T m = r \right\}$$

denote all possible exponents for monomials of degree r, where for $m \in \mathbb{I}_n(r)$, $z^m := \prod_{i=1}^n (z_i)^{m_i}$. Note that $|\mathbb{I}_n(r)| = \binom{n+r-1}{r}$. For $m \in \mathbb{I}_n(r)$, define

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Theorem (Bomze and DeKlerk, 2002)

For $r \geq 0$, $\mathcal{L}^r_n = \{ M \mid F_m \bullet M \geq 0 \;\forall m \in \mathbb{I}_n(r+2) \}$.

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For $r \geq 0$, $\mathcal{L}_n^r = \{M \mid F_m \bullet M \geq 0 \ \forall m \in \mathbb{I}_n(r+2)\}$.

Since $\mathcal{L}_n^r \subset \mathcal{C}_n^*$ is polyhedral, this implies

$$\mathrm{Co}\{mm^T \mid m \in \mathbb{I}_n(r+2)\} \subset \mathcal{C}_n \subset \mathrm{Co}\{F_m \mid m \in \mathbb{I}_n(r+2)\}.$$

Left side is obvious; right side is not.

Pena, Vera and Zuluaga (2007) define another hierarchy of inner approximations $\{\mathcal{Q}_n^r\}$ of $\mathcal{C}_n^*$. Letting $z = (x \circ x)$, for $p \in \mathbb{I}_n(r)$ we then have $z^p = x^{2p}$.

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$$\mathcal{Q}_n^r := \left\{ M \,\middle|\, \exists \{M_p\}_{p \in \mathbb{I}_n(r)} \subseteq \mathcal{D}_n^*, \left(\sum_{i=1}^n z_i\right)^r z^T M z = \sum_{p \in \mathbb{I}_n(r)} z^p z^T M_p z \right\}.$$

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It is then easy to check that for any $r \in \mathbb{Z}_+$,

$$\mathcal{L}_n^r \subseteq \mathcal{Q}_n^r \subseteq \mathcal{K}_n^r$$

and $\mathcal{Q}_n^r \subsetneq \mathcal{K}_n^r$ for $r > 1$.

Outline

1. Copositive and Completely Positive Matrices
   - Background
   - Applications in Optimization
2. Separating Doubly Nonnegative and Completely Positive matrices
   - The 5x5 case
   - Separation for matrices with block structure
   - Applications
3. Approximations of the Copositive and Completely Positive cones
   - Approximation hierarchies for CoP
   - Approximation hierarchies for CP
4. Open Problems

Natural to consider approximations for $\mathcal{C}_n$ based on symmetric tensors.

- Let $\mathcal{M}_n^r$ be the set of r-degree real-valued tensors, where each coordinate index takes on the values $1, 2, \ldots, n$. ($\mathcal{M}_n^1$ are vectors in $\mathbb{R}^n$, and $\mathcal{M}_n^2$ are $n \times n$ real matrices.)

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- Let $\mathbb{N}_r(n)$ denote the set of indexing vectors for elements of $\mathcal{M}_n^r$,

$$\mathbb{N}_r(n) := \{\alpha \in \mathbb{Z}_+^n \mid 1 \leq \alpha_i \leq n, i = 1, ..., r\}.$$

If $Z \in \mathcal{M}_n^r$ and $\alpha \in \mathbb{N}_r(n)$, $Z[\alpha] \in \mathbb{R}$ denotes the element of Z at coordinate α.

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  If $Z \in \mathcal{M}_n^r$ and $\alpha \in \mathbb{N}_r(n)$, $Z[\alpha] \in \mathbb{R}$ denotes the element of Z at coordinate α.
- Z is *symmetric* if $Z[\pi(\alpha)] = Z[\alpha]$ for any permutation $\pi(\cdot)$. We use $\mathcal{S}_n^r$ to denote the set of symmetric tensors in $\mathcal{M}_n^r$.

For $\beta \in \mathbb{N}_r(n)$, and $T \in \mathcal{M}_n^{r+2}$, T^β denotes the ordinary $n \times n$ matrix obtained by fixing the first r indices of T as β. Each such matrix is a "slice" of T, and we use **Slices**(T) to denote the set of all such slices,

$$\mathbf{Slices}(T) = \{T^\beta \mid \beta \in \mathbb{N}_r(n)\}.$$

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We also define an operator **Collapse** : $\mathcal{M}_n^{r+2} \longrightarrow \mathcal{M}_n^2$ as:

$$\textbf{Collapse}(T) = \sum_{\beta \in \mathbb{N}_r(n)} T^\beta.$$

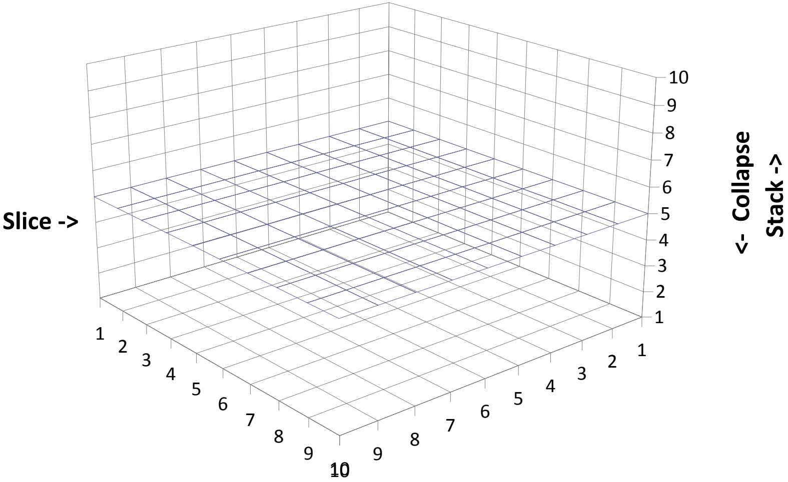

Figure: Illustration of tensor operations for $r = 1$

Consider a vector $x \in \mathbb{R}^n_+$ with $e^T x = 1$, and the "outer product" tensor $Z \in \mathcal{S}^{r+2}_n$ with

$$Z[\alpha] = \prod_{i=1}^{r+2} x_{\alpha_i}.$$

For example, if $r = 1$, then $Z[(1,2,4)^T] = x_1 x_2 x_4$, $Z[(2,3,2)^T] = x_2^2 x_3$.

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For example, if $r = 1$, then $Z[(1,2,4)^T] = x_1 x_2 x_4$, $Z[(2,3,2)^T] = x_2^2 x_3$. Note **Slices**$(Z) \subset \mathcal{D}_n$ since each slice of Z is a nonnegative multiple of xx^T, and

$$\begin{aligned}
\textbf{Collapse}(Z) &= \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_r=1}^{n} \left(\prod_{i=1}^{r} x_{\alpha_i} \right) \\
&= (e^T x)^r xx^T \\
&= xx^T.
\end{aligned}$$

Suggests defining the following cones for integer $r \geq 0$:

$$\mathcal{T}_n^r := \left\{ X = \textbf{Collapse}(Z) \,\middle|\, Z \in \mathcal{S}_n^{r+2}, \textbf{Slices}(Z) \subseteq \mathcal{N}_n \right\},$$
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Then easy to show that

$$\begin{aligned}
\mathcal{C}_n \subseteq \cdots \subseteq \mathcal{T}_n^{r+1} \subseteq \quad \mathcal{T}_n^r \quad &\subseteq \cdots \subseteq \mathcal{T}_n^1 \subseteq \mathcal{T}_n^0 := \mathcal{N}_n \\
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Theorem (Dong 2010)

For any nonnegative integer r, the cones $\mathcal{L}_n^r$ and $\mathcal{T}_n^r$ are dual to one another, as are the cones $\mathcal{Q}_n^r$ and $\mathcal{TD}_n^r$,

$$\left(\mathcal{L}_n^r\right)^* = \mathcal{T}_n^r, \qquad \left(\mathcal{T}_n^r\right)^* = \mathcal{L}_n^r,$$
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Related result due to Laurent and Gvozdenovic (2007) shows that dual of $\mathcal{K}_n^r$ corresponds to "collapsing" semidefinite relaxation of moment matrix.

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- Computational investigations - $\mathcal{T}_n^r \cap \mathcal{S}_n^+$ looks attractive.

Thank You